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SELF-SIMILAR PROBLEMS IN ELASTODYNAMICS

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CONTENTS

This paper presents a formulation of self-similar mixed boundary-value problems of elastodynamics that is a natural extension of one already developed by the writer for elastostatic problems. By thus exposing the analytical structure that is common to both the dynamic and static problems, the existence of properties common to certain static and dynamic problems is explained, and further such properties are derived. Common features of both two-dimensional and three-dimensional problems are brought out by reducing them to Hilbert problems, directly in two dimensions and by introducing the Radon transform in three dimensions. Several applications of the theory are presented, typical problems involving the indentation of a half-space by a conical or wedge-shaped indentor, and cracks expanding under the influence of a non-uniform applied stress. More difficult problems, that have not before been formulated, include dynamic indentation problems with adhesion, and problems of cracks expanding on interfaces between dissimilar materials. A method of solution of such problems is presented and an example of each type is worked out in detail.

The method of analysis hinges upon representations of the solutions of 'unmixed' self-similar problems for half-spaces, which are obtained by use of an alternative to Cagniard's technique whose application is routine, even for an anisotropic half-space. The representations provide more general solutions of the unmixed problems than were available previously. The main singularities, or 'arrivals', of the stress fields are extracted from the representations; these expressions are new and should be useful for certain problems in seismology. It is predicted, for instance, that a crack expanding on an interface can generate a 'conical wave', that is, a region in which the singularity has a logarithmic component as well as a step function, even in its P-wave arrival, which could not occur for a crack in a homogeneous medium.

The properties of the equations of elastodynamics that are employed are that they are linear, homogeneous and self-adjoint and the methods that are developed are equally applicable to any other system with these properties.

1. Introduction

The main motivation behind the present work was to formulate and solve a class of mixed boundary-value problems in elastodynamics, of which a representative is provided by a crack expanding uniformly from a point under the action of a prescribed stress. Before this investigation, the most general three-dimensional solution that was known was that of Burridge & Willis (1969), for a crack of elliptical shape expanding under a uniform stress in an anisotropic medium. The problem was solved, quite simply, by guessing the form of the relative displacement of the crack faces and then demonstrating its correctness. This relative displacement has a simple form, independent of the speed of the crack, and is the same as is obtained in the limit of a stationary crack. The static limit had been investigated more generally by Willis (1968), for polynomial loading of the crack, again by guessing the form of the relative displacement, but confirming its validity by an entirely different set of manipulations.

Recently, Willis (1971 *a*, 1972) has developed a much more systematic approach to static problems, which produces the former results in a rational way, and which also facilitates the solution of some much more difficult problems involving a crack on an interface between dissimilar materials. The present work comprises a generalization of this approach, to a class of dynamic problems whose solutions are homogeneous functions of the time t and position x; the problem solved by Burridge & Willis (1969) falls into this category since the stresses are homogeneous functions of degree zero in (t, x). The formulation of the dynamic problems reduces to that already developed in the static case by the simple device of letting the density of the material tend to zero and a unified approach, which exposes the common structural features of both the dynamic and static problems, is thereby obtained. It follows immediately that the static result of Willis (1968) can be generalized to the dynamic case. Also, at a deeper level, it becomes possible for the first time to study dynamically expanding cracks on interfaces. Solutions of such problems are expressed in terms of the solution of a certain Hilbert problem, which can be solved exactly in the static limit and for which a procedure for developing systematic approximations to its solution is outlined in the dynamic case.

Dual problems, typically involving the dynamic indentation of a half-space, either with or without adhesion, can be formulated and solved similarly, and in fact, are discussed before crack problems in the text because they are slightly easier to describe. Two-dimensional problems can be approached by the same basic technique, and lead to a formulation of mixed boundary-value problems that requires the solution of Hilbert problems, very closely related to the corresponding static problems set out by Muskhelishvili (1953). Again, as in the static work of Willis (1971 *a*, 1972), the close relation between two- and three-dimensional problems is brought out, the solution of both being expressible in terms of the solution of the same Hilbert problem. For problems that are axisymmetric, the three-dimensional formulations degenerate immediately to yield equations of Abel type. Thus, many of the static crack problems discussed by Lowengrub & Sneddon (1970), for instance, can be generalized to self-similarly growing cracks and, in addition, many problems of a truly three-dimensional nature can be solved. The formulation thus unifies a large part of elastostatics and elastodynamics, both two-dimensional and three-dimensional, though the self-similar property that is assumed precludes any possible extension to bodies of finite extent, for example, as this would introduce a characteristic length.

In the course of the work, new formulations of the traction and displacement boundary-value problems for an anisotropic half-space are given, which are used as points of departure for solving mixed problems. They yield immediate solutions of the unmixed problems that are more general than any given before, but worked examples are confined, in the main, to the more difficult mixed problems in the present work. In both the two-dimensional and three-dimensional problems, the main singularities, or 'arrivals', in the stress fields are investigated. The resulting expressions are more general than any already available, even for 'Lamb's problem', and should be useful for certain applications in seismology.

It is convenient here to introduce the notation that will be used throughout the paper. Both matrix and tensor notations are used, as the context demands. The summation convention is employed, with Latin suffixes taking the values 1, 2, 3 and Greek suffixes the values 1, 2 only. Relative to a set of cartesian axes, the displacement vector has components $u_i(t, x)$. The stress tensor has components $\sigma_{ij}(t, x)$ and is related to the displacement gradients through the generalized Hooke's law

$$\sigma_{ij} = c_{ijkl} u_{k,l}, \tag{1.1}$$

where $_{,l}$ denotes $\partial/\partial x_l$ and c_{ijkl} is the usual tensor of elastic moduli, with the symmetries

$$c_{ijkl} = c_{jikl} = c_{klij}. \tag{1.2}$$

Substitution of (1.1) into the equations of motion, without body forces, gives the self-adjoint, totally hyperbolic system

$$c_{ijkl} u_{k,lj} = \rho \ddot{u}_i, \tag{1.3}$$

where ρ is the density of the medium and a superposed dot denotes $\partial/\partial t$. When matrix notation is used, $u(t, x)$ denotes the column vector with components $u_i(t, x)$ and the equations of motion (1.3) are written

$$K(\partial_t, \nabla)\, u = 0, \tag{1.4}$$

where $K(\omega, \xi)$ is a matrix with components

$$K_{ik}(\omega, \xi) = c_{ijkl} \xi_j \xi_l - \rho \omega^2 \delta_{ik}. \tag{1.5}$$

Although the work is expressed entirely in terms of the above equations of elastodynamics, the essential properties of these equations are that they are linear, homogeneous and self-adjoint and the approach that is developed could be applied to any other system with these properties.

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2. The homogeneous traction boundary-value problem

2.1. *The three-dimensional problem*

Let tractions $T(t, x_\alpha)$ be applied, for time $t > 0$, to the surface $x_3 = 0$ of the half-space $x_3 > 0$. The traction vector $T(t, x_\alpha)$ is homogeneous of degree n in (t, x_α), so that

$$T(t, x_\alpha) = t^n T(y) \quad (t > 0), \tag{2.1.1}$$

where

$$y = (y_1, y_2), \quad y_1 = x_1/t, \quad y_2 = x_2/t. \tag{2.1.2}$$

The displacement field $u(t, x)$ which $T(t, x_\alpha)$ generates within the half-space is clearly homogeneous of degree $n+1$ in (t, x). It satisfies the differential equation

$$K(\partial_t, \nabla)\, u = 0 \quad (x_3 > 0), \tag{2.1.3}$$

and the initial and boundary conditions

$$u(0, x) = \dot{u}(0, x) = 0 \quad (x_3 \geqslant 0), \tag{2.1.4}$$

$$\left.\begin{aligned} C(\nabla)\, u(t, x) &\to T(t, x_\alpha) \quad \text{as} \quad x_3 \to 0, \\ u(t, x) &\to 0 \quad \text{as} \quad |x| \to \infty, \end{aligned}\right\} \tag{2.1.5}$$

where the matrix $C(\xi)$ has components

$$C_{ik}(\xi) = c_{i3kl}\, \xi_l. \tag{2.1.6}$$

The second of conditions (2.1.5) implies, of course, a restriction on $T(t, x_\alpha)$ as $(x_1^2 + x_2^2) \to \infty$; we will, in fact, assume below that the function $T(y)$ defined in (2.1.1) is integrable along any line $\eta_\alpha y_\alpha = p$.

A formal solution of the problem defined above may be obtained by taking Fourier transforms. It will be apparent, however, that the Fourier transform so constructed is integrable over any bounded domain and has at most polynomial behaviour at infinity. It is thus a temperate distribution (Hörmander 1963) and hence its inverse $u(t, x)$ is an actual solution in the sense of distributions which, moreover, is unique within the space of temperate distributions. We define the Fourier transform $\tilde{u}(\omega, \xi_\alpha, x_3)$ of $u(t, x)$ as

$$\tilde{u}(\omega, \xi_\alpha, x_3) = (2\pi)^{-\frac{3}{2}} \int_0^\infty \mathrm{d}t \iint \mathrm{d}x_1\, \mathrm{d}x_2\, u(t, x) \exp\{\mathrm{i}(\omega t + \xi_\alpha x_\alpha)\}. \tag{2.1.7}$$

It will be convenient to let ω be complex but to keep ξ_1, ξ_2 real. For fixed ξ_1, ξ_2, x_3, equation (2.1.7) defines $\tilde{u}(\omega, \xi_\alpha, x_3)$ as a function of ω which is analytic in the upper half-plane so that the correct inverse of (2.1.7) is

$$u(t, x) = (2\pi)^{-\frac{3}{2}} \iint \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}} \mathrm{d}\omega\, \tilde{u}(\omega, \xi_\alpha, x_3) \exp\{-\mathrm{i}(\omega t + \xi_\alpha x_\alpha)\}. \tag{2.1.8}$$

Equation (2.1.8) defines a $u(t, x)$ which is zero for $t < 0$. If the existence of $u(t, x)$ is assumed, the analytic property of $\tilde{u}(\omega, \xi_\alpha, x_3)$ follows. However, our object is to find $u(t, x)$ and this will be done by constructing $\tilde{u}(\omega, \xi_\alpha, x_3)$ directly. The analytic property of the $\tilde{u}(\omega, \xi_\alpha, x_3)$ so constructed should therefore be checked directly, as should the claim that $\tilde{u}(\omega_1 + 0\mathrm{i}, \xi_\alpha, x_3)$ is a temperate distribution in the real variables ω_1, ξ_1, ξ_2, for each fixed x_3; these properties will, however, be evident below and will not be discussed further.

To proceed now to details, equations (2.1.3) and (2.1.7) imply that

$$K(\omega, \xi_1, \xi_2, \mathrm{i}\partial_3)\,\tilde{u}(\omega, \xi_\alpha, x_3) = 0 \quad (x_3 > 0). \tag{2.1.9}$$

The general solution of the ordinary differential equations (2.1.9) may be found by seeking solutions of the form

$$\tilde{u} = v \exp\{-\mathrm{i}\xi_3 x_3\}, \tag{2.1.10}$$

in which the column vector v is independent of x_3. The expression (2.1.10) satisfies (2.1.9) if

$$K(\omega, \xi)\, v = 0, \tag{2.1.11}$$

where ξ has components (ξ_1, ξ_2, ξ_3). This equation has solutions if

$$|K(\omega, \xi)| = 0. \tag{2.1.12}$$

Equation (2.1.12) has, for each fixed (ω, ξ_1, ξ_2) six roots $\xi_3 = \xi_3^N(\omega, \xi_\alpha)$ $(N = 1, 2, \ldots, 6)$, which may either be real or may occur in complex conjugate pairs when ω, ξ_1, ξ_2 are all real. Correspondingly, the general solution of (2.1.9) may be expressed as

$$\tilde{u}(\omega, \xi_\alpha, x_3) = \sum_{N=1}^{6} [\operatorname{adj} K(\omega, \xi^N)]\, b^N(\omega, \xi_\alpha) \exp\{-\mathrm{i}\xi_3^N x_3\} \tag{2.1.13}$$

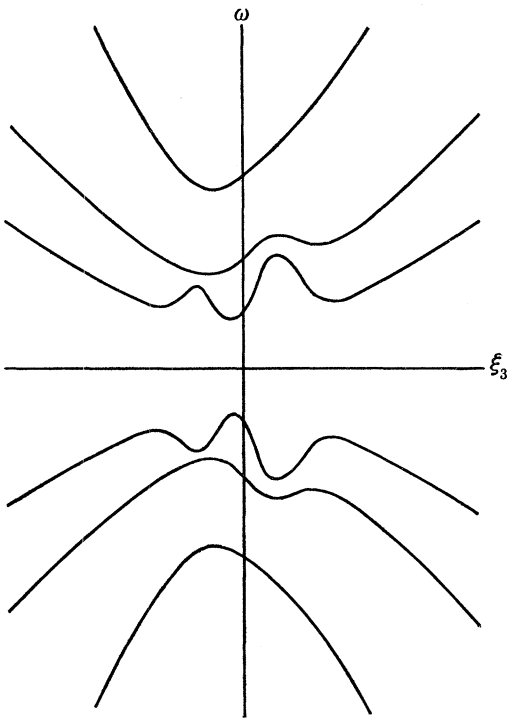

FIGURE 1. A possible set of curves of ω against ξ_3, for fixed ξ_1 and ξ_2, obtained from equation (2.1.12).

where ξ^N denotes the vector $(\xi_1, \xi_2, \xi_3^N(\omega, \xi_\alpha))$ and the b^N are to be determined from the boundary conditions (2.1.5). First, to satisfy $(2.1.5)_2$, values of N for which $\xi_3^N(\omega, \xi_\alpha)$ have positive imaginary part when ω lies just above the real axis must be rejected. To decide which values of N are relevant it is necessary to discuss briefly the Riemann surface of the algebraic function $\xi_3(\omega, \xi_\alpha)$ which is defined by equation (2.1.12); the argument to follow is very closely related to one given by Burridge (1970). The function $\xi_3(\omega, \xi_\alpha)$ is single-valued if ω is allowed to range over the six sheets

Σ^N $(N = 1, 2, \ldots, 6)$ of its Riemann surface, taking the values $\xi_3^N(\omega, \xi_\alpha)$ on Σ^N. The sheets are connected across appropriate lines joining the branch points of the functions $\xi_3^N(\omega, \xi_\alpha)$. The most important branch points are those on the real axis in the complex ω-plane and these may be discussed by reference to figure 1. This shows a possible set of curves, defined by equation (2.1.12), for fixed ξ_1, ξ_2, when ω and ξ_3 are real: that there are six real curves follows from the assumption that equations (2.1.3) are totally hyperbolic, and they are symmetric relative to the line $\omega = 0$ because equation (2.1.12) contains only ω^2. Each curve has an equation of the form

$$\omega(\xi) = \pm(\xi_1^2+\xi_2^2+\xi_3^2)^{\frac{1}{2}}\,c(\zeta), \tag{2.1.14}$$

where
$$\zeta = (\xi_1, \xi_2, \xi_3)/(\xi_1^2+\xi_2^2+\xi_3^2)^{\frac{1}{2}} \tag{2.1.15}$$

is a unit vector in the direction of ξ and $c(\zeta)$ is one of the possible speeds of plane waves propagating in this direction.

The roots $\xi_3^N(\omega_0, \xi_\alpha)$, when they are real, are the intersections of the line

$$\omega = \omega_0 \tag{2.1.16}$$

with the curves (2.1.12) shown in figure 1. If $|\omega_0|$ is sufficiently large, there are six real roots $\xi_3^N(\omega_0, \xi_\alpha)$. Three of these roots correspond to intersections at which $\partial\omega/\partial\xi_3 > 0$ and three to intersections at which $\partial\omega/\partial\xi_3 < 0$. The three of the former type will be assigned to the Riemann surfaces Σ^N $(N = 1, 2, 3)$ and the three of the latter type to Σ^N $(N = 4, 5, 6)$. Branch points occur at points of tangency of (2.1.16) with the curves (2.1.12), at which each of a certain pair of roots, $\xi^M(\omega, \xi_\alpha)$ and $\xi^N(\omega, \xi_\alpha)$ say, will cease to be real but will become complex conjugates. A branch line, joining Σ^N to Σ^M may now be introduced, by joining the point (ω_0, ξ_3) to $(-\omega_0, \xi_3)$, at which the same phenomenon will occur. Similar branch lines may be drawn between all points of tangency for $\omega_0 > 0$ with the corresponding points with $\omega_0 < 0$. Parts of these may be redundant at points at which two roots again become real (at maxima of the curves (2.1.12) if $\omega_0 > 0$), but the construction has now defined the most important branch lines of the Riemann surface. On the sheets Σ^N $(N = 1, 2, 3)$, if now ω moves from right to left just above the real axis, the corresponding root $\xi_3^N(\omega, \xi_\alpha)$ is real when ω is large and positive, and remains real and decreasing with ω until a branch point is reached. Thereafter, because ω passes to the right of the branch point (which is assumed to be simple), $\xi_3^N(\omega, \xi_\alpha)$ turns right and hence has positive imaginary part. It therefore has positive imaginary part in the whole of the upper half of Σ^N since otherwise it would have zero imaginary part at some point and this can only occur when ω is real. Similarly, the branches ξ_3^N $(N = 4, 5, 6)$ have negative imaginary part in the whole of the upper half of Σ^N $(N = 4, 5, 6)$. Branch points could also occur off the real ω-axis, but these could only link Σ^M to Σ^N, where M and N both lie either in the range $(1, 2, 3)$ or $(4, 5, 6)$.

It is now clear that only the values $N = 4, 5, 6$ can be admitted in the sum (2.1.13) if condition $(2.1.5)_2$ is to hold. It may be noted in passing that equation (2.1.8) now represents $u(t, x)$ as a superposition of plane waves which, when ξ_3^N is real, have 3-components of group velocity which are positive, so that energy is propagated away from the surface (Lighthill 1960). The condition $(2.1.5)_1$ now implies that

$$\sum_{N=4}^{6} C(\xi^N)\,\mathrm{adj}\,K(\omega, \xi^N)\,b^N(\omega, \xi_\alpha) = \mathrm{i}\,\tilde{T}(\omega, \xi_\alpha), \tag{2.1.17}$$

where
$$\tilde{T}(\omega, \xi_\alpha) = (2\pi)^{-\frac{3}{2}}\int_0^\infty \mathrm{d}t \iint \mathrm{d}x_1\,\mathrm{d}x_2\,T(t, x_\alpha)\exp\{\mathrm{i}(\omega t+\xi_\alpha x_\alpha)\} \tag{2.1.18}$$

is the Fourier transform of $T(t, x_\alpha)$. Equation (2.1.17) can be satisfied by taking

$$b^N(\omega, \xi_\alpha) = \mathrm{i}\left[\sum_{M=4}^{6} C(\xi^M)\,\mathrm{adj}\,K(\omega, \xi^M)\right]^{-1} \tilde{T}(\omega, \xi_\alpha), \tag{2.1.19}$$

independently of N. Hence,

$$\tilde{u}(\omega, \xi_\alpha, x_3) = (2\pi)^{\frac{3}{2}} \sum_{N=4}^{6} \exp\{-\mathrm{i}\xi_3^N x_3\}\, B^N(\omega, \xi_\alpha)\, \tilde{T}(\omega, \xi_\alpha), \tag{2.1.20}$$

where
$$B^N(\omega, \xi_\alpha) = (2\pi)^{-\frac{3}{2}}\mathrm{i}\,\mathrm{adj}\,K(\omega, \xi^N)\left[\sum_{M=4}^{6} C(\xi^M)\,\mathrm{adj}\,K(\omega, \xi^M)\right]^{-1}. \tag{2.1.21}$$

Before proceeding with the inversion of (2.1.20), we remark that, since $T(t, x_\alpha)$ is homogeneous of degree n, $\tilde{T}(\omega, \xi_\alpha)$ is homogeneous of degree $-(n+3)$ in (ω, ξ_α) and, further, is analytic in the upper half of the complex ω-plane for each real ξ_α. Also, from (2.1.21), $B^N(\omega, \xi_\alpha)$ may be recognized as a homogeneous function of degree -1 in (ω, ξ_α), since $\xi_3^N(\omega, \xi_\alpha)$ is homogeneous of degree 1. In view of these properties, it will prove more convenient to calculate first $\partial_t^{(n+2)} u(t, x)$, whose Fourier transform is (2.1.20) multiplied by $(-\mathrm{i}\omega)^{n+2}$. Thus, we have

$$\partial_t^{(n+2)} u(t, x) = (-\mathrm{i})^{n+2} \sum_{N=4}^{6} \iint \mathrm{d}\xi_1\,\mathrm{d}\xi_2 \int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}} \mathrm{d}\omega\,\omega^{n+2} B^N(\omega, \xi_\alpha)\,\tilde{T}(\omega, \xi_\alpha)\exp\{-\mathrm{i}(\omega t + \xi^N . x)\} \tag{2.1.22}$$

which may be evaluated, in the sense of distributions, as

$$\partial_t^{(n+2)} u(t, x) = (-\mathrm{i})^{n+2} \sum_{N=4}^{6} \lim_{\epsilon\to 0} \iint \mathrm{d}\xi_1\,\mathrm{d}\xi_2 \int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}} \mathrm{d}\omega\,\omega^{n+2} B^N(\omega, \xi_\alpha)\,\tilde{T}(\omega, \xi_\alpha) \times \exp\{-\mathrm{i}(\omega t + \xi^N . x - \mathrm{i}\epsilon|\xi|)\}, \tag{2.1.23}$$

where $|\xi| = (\xi_1^2 + \xi_2^2)^{\frac{1}{2}}$. The integrals in (2.1.23) may now be simplified by a method which has previously been used in a less general context by Eason (1966); we set

$$\eta_\alpha = \xi_\alpha/|\xi|, \quad \Omega = \omega/|\xi|, \tag{2.1.24}$$

and transform to polar coordinates in (ξ_1, ξ_2)-space. The homogeneous properties of the functions B^N and $\tilde{T}$ then reduce the resulting integrals to the form

$$\partial_t^{(n+2)} u(t, x) = (-\mathrm{i})^{n+2} \sum_{N=4}^{6} \lim_{\epsilon\to 0} \oint_{|\eta|=1} \mathrm{d}s \int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}} \mathrm{d}\Omega\,\Omega^{n+2} B^N(\Omega, \eta_\alpha)\,\tilde{T}(\Omega, \eta_\alpha) \times \int_0^\infty \mathrm{d}\,|\xi| \exp\{-\mathrm{i}\,|\xi|\,(\Omega t + \xi^N(\Omega, \eta) . x - \mathrm{i}\epsilon)\}$$

or, upon integrating with respect to $|\xi|$,

$$\partial_t^{(n+2)} u(t, x) = -(-\mathrm{i})^{n+1} \sum_{N=4}^{6} \lim_{\epsilon\to 0} \oint_{|\eta|=1} \mathrm{d}s \int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}} \mathrm{d}\Omega\, \frac{\Omega^{n+2} B^N(\Omega, \eta_\alpha)\,\tilde{T}(\Omega, \eta_\alpha)}{\Omega t + \eta_\alpha x_\alpha + \xi_3^N(\Omega, \eta)\, x_3 - \mathrm{i}\epsilon}. \tag{2.1.25}$$

The integral with respect to Ω may now be calculated by closing the contour in the upper half of the complex Ω-plane and employing Cauchy's theorem. The functions $B^N(\Omega, \eta_\alpha)$ and $\tilde{T}(\Omega, \eta_\alpha)$ are analytic in the upper half of the complex Ω-plane and the only contributions to the integral come from the zeros of the functions

$$\phi^N(\Omega) = \Omega t + \eta_\alpha x_\alpha + \xi_3^N(\Omega, \eta)\, x_3 - \mathrm{i}\epsilon, \tag{2.1.26}$$

if there are any such zeros in the upper half-plane. If $t \leqslant 0$ there are none, since the imaginary part of $\xi_3^N(\Omega, \eta)$ is negative in the upper half-plane. Therefore, there is a $t_N \geqslant 0$ such that $\phi^N(\Omega)$

has no zero in the upper half-plane for $t \leqslant t_N$. The value of t_N may be estimated as follows. Let Ω^N satisfy the equation

$$\Omega^N t + \eta_\alpha x_\alpha + \xi_3^N(\Omega^N, \eta)\, x_3 = 0. \tag{2.1.27}$$

Then, since ϵ is arbitrarily small, the root Ω of (2.1.26) is given asymptotically as

$$\Omega = \Omega^N + \epsilon\Omega_1,$$

where Ω_1 satisfies the equation $\quad \Omega_1(t + \xi^N_{3,\Omega}(\Omega^N, \eta)\, x_3) = \mathrm{i}. \quad$ (2.1.28)

If Ω^N lies in the upper half-plane, then so does Ω since ϵ is small. For small t, however, equation (2.1.27) has a real root and then Ω lies in the upper half-plane only if

$$t + \xi^N_{3,\Omega}(\Omega^N, \eta)\, x_3 > 0. \tag{2.1.29}$$

For any particular medium, this inequality could be studied directly. However, in the general case, when $x_3 = 0$, $\phi^N(\Omega) = 0$ when $\Omega = (-\eta_\alpha x_\alpha + \mathrm{i}\epsilon)/t$, which lies in the upper half-plane for all $t > 0$ and is singular as $t \to 0$. It can now be shown that there is always a transition when $|\Omega| \to \infty$. If Ω^N is real and $|\Omega^N|$ is large, then

$$\xi_3^N(\Omega^N, \eta) \sim -\Omega^N/c(0,0,1)$$

and

$$\xi^N_{3,\Omega}(\Omega^N, \eta) \sim -1/c(0,0,1),$$

where $c(0,0,1)$ is one of the speeds with which plane waves may propagate in the 3-direction. The inequality (2.1.29) now holds for $t > t_N$, where

$$t_N = x_3/c(0,0,1). \tag{2.1.30}$$

That this t_N is the smallest value of t for which $\phi^N(\Omega)$ has a root in the upper half-plane now follows by continuity with respect to the variable x_3.

Having established the value of t_N, application of Cauchy's theorem to (2.1.25) yields, letting $\epsilon \to 0$,

$$\partial_t^{(n+2)} u(t,x) = 2\pi(-\mathrm{i})^{n+2} \sum_{N=4}^{6} \oint_{|\eta|=1} \mathrm{d}s\, (\Omega^N)^{n+2} \frac{B^N(\Omega^N, \eta_\alpha)\, \tilde{T}(\Omega^N, \eta_\alpha)}{t + \xi^N_{3,\Omega}(\Omega^N, \eta)\, x_3} H(t - t_N), \tag{2.1.31}$$

in which Ω^N satisfies both (2.1.27) and (2.1.29) for $t > t_N$. Equation (2.1.31) is our basic result, from which $u(t,x)$ may be obtained by integration. In preparation for displaying the close structural relationship between the present solution and its two-dimensional counterpart, which will be discussed in § 2.3, we now represent $\tilde{T}(\omega, \xi_\alpha)$ as a Cauchy integral as follows. We have

$$\tilde{T}(\omega, \xi_\alpha) = (2\pi)^{-\frac{3}{2}} \int_0^\infty \mathrm{d}t \exp\{\mathrm{i}\omega t\} \iint \mathrm{d}x_1\, \mathrm{d}x_2 \exp\{\mathrm{i}\xi_\alpha x_\alpha\}\, T(t, x_\alpha),$$

which may be reduced by transforming to the variables

$$y_\alpha = x_\alpha/t$$

and using (2.1.1) to give

$$\tilde{T}(\omega, \xi_\alpha) = (2\pi)^{-\frac{3}{2}} \int_0^\infty t^{n+2}\, \mathrm{d}t \iint \mathrm{d}y_1\, \mathrm{d}y_2\, T(y) \exp\{\mathrm{i}t(\omega + \xi_\alpha y_\alpha)\}. \tag{2.1.32}$$

Integration with respect to t (for $\mathrm{Im}\,(\omega) > 0$) now yields

$$\tilde{T}(\omega, \xi_\alpha) = (2\pi)^{-\frac{3}{2}} (-\mathrm{i})^{n+1} \partial_\omega^{(n+2)} \iint \mathrm{d}y_1\, \mathrm{d}y_2 \frac{T(y)}{\omega + \xi_\alpha y_\alpha},$$

which may be reduced finally by integrating along the line

$$\xi_\alpha y_\alpha = p$$

to give
$$\tilde{T}(\omega, \xi_\alpha) = (2\pi)^{-\frac{3}{2}}(-\mathrm{i})^{n+1}\,\partial_\omega^{(n+2)}\int \frac{\check{T}(p,\xi_\alpha)\,\mathrm{d}p}{\omega+p}, \tag{2.1.33}$$

where
$$\check{T}(p,\xi_\alpha) = \iint \mathrm{d}y_1\,\mathrm{d}y_2\,T(y)\,\delta(p-\xi_\alpha y_\alpha) \tag{2.1.34}$$

is the Radon transform of $T(y)$. Some important properties of the Radon transform have been obtained by Ludwig (1966), by exploiting its close relationship to the Fourier transform; those relevant to the present work are summarized in the appendix.

By substituting (2.1.33) into (2.1.31), we now obtain

$$\partial_t^{(n+2)}u(t,x) = -(2\pi)^{\frac{1}{2}}\sum_{N=4}^{6}\oint_{|\eta|=1}\mathrm{d}s\,(\Omega^N)^{n+2}\frac{B^N(\Omega^N,\eta_\alpha)\,F^{(n+2)}(-\Omega^N,\eta)}{t+\xi_3^N(\Omega^N,\eta)\,x_3}H(t-t_N), \tag{2.1.35}$$

where
$$F^{(n+2)}(z,\eta) = \frac{1}{2\pi\mathrm{i}}\,\partial_z^{(n+2)}\int\frac{\check{T}(p,\eta_\alpha)\,\mathrm{d}p}{p-z}. \tag{2.1.36}$$

It can be proved that the representation (2.1.35) defines a real function $\partial_t^{(n+2)}u(t,x)$ by replacing η by $-\eta$. It is evident that, if $\xi_3^N(\omega,\eta)$ $(N = 4, 5, 6)$ is a root of the equation

$$|K(\omega,\eta_\alpha,\xi_3)| = 0,$$

then $-\overline{\xi_3^N(\omega,\eta)}$ is a root of
$$|K(-\overline{\omega}, -\eta_\alpha, \xi_3)| = 0,$$

where the superposed bar denotes the complex conjugate. Therefore, if ξ_3^N is complex, $\xi_3^M(-\overline{\omega},-\eta) = -\overline{\xi_3^N(\omega,\eta)}$ for some M not necessarily equal to N but still in the range (4, 5, 6). Analytic continuation now shows that this result remains true when ξ_3^N is real. Hence, by taking minus the complex conjugate of (2.1.27), it is obtained that

$$-\overline{\Omega^N}t-\eta_\alpha x_\alpha+\xi_3^M(-\overline{\Omega^N},-\eta)\,x_3 = 0,$$

so that
$$\Omega^M(-\eta) = -\overline{\Omega^N(\eta)} \quad\text{and}\quad \xi_3^M(\Omega^M(-\eta),-\eta) = -\overline{\xi_3^N(\Omega^N(\eta),\eta)}.$$

That the right side of (2.1.35) is its own complex conjugate now follows upon use of the even property (A 2) of the Radon transform.

The $n+2$ integrations with respect to t which are required to produce $u(t,x)$ cannot be performed explicitly but it is possible to find a simple expression for $\partial_t^{(n+1)}u(t,x)$. This is done by introducing inside the contour integral the variable Ω^N in place of t. By differentiating equation (2.1.27), it is obtained that

$$\frac{\mathrm{d}\Omega^N}{\Omega^N} = -\frac{\mathrm{d}t}{t+\xi^N_{3,\Omega}(\Omega^N,\eta)\,x_3}.$$

Therefore,

$$\partial_t^{(n+1)}u(t,x) = -(2\pi)^{\frac{1}{2}}\sum_{N=4}^{6}\oint_{|\eta|=1}\mathrm{d}s\int_{\Omega\ (t,x,\eta)}^{\Omega^N(t_N,x,\eta)}\mathrm{d}\Omega\,\Omega^{n+1}B^N(\Omega,\eta_\alpha)\,F^{(n+2)}(-\Omega,\eta), \tag{2.1.37}$$

which may be simplified a little furthe1 by remembering that $|\Omega^N(t_N,x,\eta)|$ is infinite. The integral with respect to Ω is thus evaluated along the path defined by equation (2.1.27), which begins at $\Omega^N(t,x,\eta)$ and ends at infinity, and always lies in the upper half-plane. By virtue of the fact that the integrand is analytic in the upper half-plane and is $O(1/\Omega^3)$ as $|\Omega|\to\infty$, the integral can in fact be evaluated along any path from $\Omega^N(t,x,\eta)$ to infinity which lies in the upper half-plane.

2.2. *The arrivals for the three-dimensional problem*

The singularities, or arrivals, of the field $u(t, x)$ are of particular interest; they arise physically from the arrivals of disturbances travelling at one of the possible wave speeds. A complete analysis of all possibilities would be very laborious and attention will be restricted below only to the simplest cases. In particular, the loading $T(t, x_\alpha)$ will be assumed to be zero outside a surface $S(t)$ which expands uniformly from the origin for $t \geqslant 0$. Further, the velocity of each point of the boundary $\partial S(t)$ of $S(t)$ will be assumed to be smaller than any of the possible speeds with which plane waves (including the Rayleigh wave) may propagate in the direction normal to $\partial S(t)$. Such surfaces $S(t)$ will be termed 'subsonic'; they will feature prominently in later sections of this work.

For subsonic loading, as defined above, it is clear physically that body-wave arrivals will appear at the point x, at the times at which plane waves passing through the origin at time $t = 0$ would first meet x. These times are related in a simple way to the geometry of the 'slowness surfaces' of the waves (Hearmon 1961) and the derivation to follow is designed to highlight this feature. It is convenient to start from the representation (2.1.25). It is easy to verify, by the use of an argument like the one employed at the end of the preceding section, that the integral over the negative half of the cylinder $|\eta| = 1$, $-\infty < \Omega < \infty$ is the complex conjugate of that over the positive half, so that (2.1.25) can be rewritten as

$$\partial_t^{(n+2)} u(t, x) = \mathrm{Re}\left\{2(-\mathrm{i})^{n+1} \sum_{N=4}^{5} \lim_{\epsilon\to 0} \oint_{|\eta|=1} \mathrm{d}s \int_{0\mathrm{i}}^{\infty+0\mathrm{i}} \mathrm{d}\Omega \, \frac{\Omega^{n+2} B^N(\Omega, \eta_\alpha)\, \tilde{T}(\Omega, \eta_\alpha)}{\Omega t + \eta_\alpha x_\alpha + \xi_3^N(\Omega, \eta)\, x_3 - \mathrm{i}\epsilon}\right\}, \tag{2.2.1}$$

where the symbol Re denotes 'the real part of'. The range of integration with respect to Ω may now be split into two parts: that for which $\xi_3^N(\Omega, \eta)$ is real ($\Omega > \Omega_1(\eta)$, say) and that for which it is complex ($\Omega < \Omega_1(\eta)$). Singularities of the integrand will occur in the former range, if the real part of the denominator changes sign, and these can lead to 'arrivals'. To demonstrate this, the integral I over that part of the cylinder for which $\Omega > \Omega_1(\eta)$ is first transformed by replacing the variable Ω by the variable $\xi_3^N(\Omega, \eta)$ to give

$$I(t, x) = \lim_{\epsilon\to 0} \oint_{|\eta|=1} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}\xi_3 \frac{\partial\omega}{\partial\xi_3}(\eta, \xi_3)\, H\left(-\frac{\partial\omega}{\partial\xi_3}\right) \frac{\omega^{n+2} B^N(\omega, \eta_\alpha)\, \tilde{T}(\omega, \eta_\alpha)}{\omega t + \eta_\alpha x_\alpha + \xi_3 x_3 - \mathrm{i}\epsilon}, \tag{2.2.2}$$

where $\omega = \omega(\eta, \xi_3)$ is one of the curves shown in figure 1, with $\omega > 0$. The integral I may now be transformed by projection onto the slowness surface S which is defined by the equation

$$\omega(\xi) = 1. \tag{2.2.3}$$

Since the integrand is homogeneous of degree -3, this gives

$$I(t, x) = \lim_{\epsilon\to 0} \int_S \frac{\mathrm{d}S}{|\nabla\omega|} \frac{\partial\omega}{\partial\xi_3} H\left(-\frac{\partial\omega}{\partial\xi_3}\right) \frac{B^N(1, \xi_\alpha)\, \tilde{T}(1, \xi_\alpha)}{t + \xi \,.\, x - \mathrm{i}\epsilon}. \tag{2.2.4}$$

A discontinuity in $I(t, x)$ occurs at the time $t = t_0$ for which the plane

$$t + \xi \,.\, x = 0 \tag{2.2.5}$$

becomes tangent to the slowness surface S. To investigate this, it suffices to consider the integral over any small neighbourhood $\mathcal{N}(\xi^0)$ of the point of tangency ξ^0. Then, since only the term $(t + \xi \,.\, x - \mathrm{i}\epsilon)^{-1}$ varies rapidly in $\mathcal{N}(\xi^0)$, the other terms in the integrand may be replaced by their values at ξ^0 and we may consider

$$\Delta I(t, x) = \frac{1}{|\nabla\omega|} \frac{\partial\omega}{\partial\xi_3} H\left(-\frac{\partial\omega}{\partial\xi_3}\right) B^N(1, \xi_\alpha^0)\, \tilde{T}(1, \xi_\alpha^0) \lim_{\epsilon\to 0} \int_{\mathcal{N}(\xi^0)} \frac{\mathrm{d}S}{t + \xi \,.\, x - \mathrm{i}\epsilon}. \tag{2.2.6}$$

The behaviour of the integral in (2.2.6) depends upon the geometry of the slowness surface near the point ξ^0. Attention will be restricted here to the simplest (and most common) possibility, that the Gaussian curvature of S is finite and non-zero at ξ^0. In this case, taking normal coordinates p, q with origin at ξ^0, the surface S is represented approximately in $\mathcal{N}(\xi^0)$ by the equation

$$(\xi-\xi^0)\,.\,x = \tfrac{1}{2}a\,|x|\,p^2+\tfrac{1}{2}b\,|x|\,q^2, \tag{2.2.7}$$

where a and b are the principal curvatures at ξ^0. Three cases may now be distinguished, $a > 0$ and $b > 0$, $a < 0$ and $b < 0$, and $ab < 0$; these are discussed below.

(i) $a > 0$, $b > 0$.

In terms of the variables p, q, the integral

$$J = \lim_{\epsilon\to 0}\int_{\mathcal{N}(\xi^0)}\frac{\mathrm{d}S}{t+\xi\,.\,x-\mathrm{i}\epsilon} \tag{2.2.8}$$

takes the form

$$J = \lim_{\epsilon\to 0}\iint\frac{\mathrm{d}p\,\mathrm{d}q}{\tfrac{1}{2}a\,|x|\,p^2+\tfrac{1}{2}b\,|x|\,q^2+t-t_0-\mathrm{i}\epsilon} \tag{2.2.9}$$

or, upon setting

$$p_1 = (\tfrac{1}{2}a\,|x|)^{\frac{1}{2}}p, \quad q_1 = (\tfrac{1}{2}b\,|x|)^{\frac{1}{2}}q, \tag{2.2.10}$$

$$J = \frac{2}{(ab)^{\frac{1}{2}}\,|x|}\lim_{\epsilon\to 0}\iint\frac{\mathrm{d}p_1\,\mathrm{d}q_1}{p_1^2+q_1^2+t-t_0-\mathrm{i}\epsilon}. \tag{2.2.11}$$

Clearly the discontinuity in J as t varies will be independent of the shape of $\mathcal{N}(\xi^0)$, and it is convenient now to choose $\mathcal{N}(\xi^0)$ so that

$$p_1^2+q_1^2 \leqslant \delta^2. \tag{2.2.12}$$

The integral in (2.2.11) may now be evaluated by transforming to polar coordinates in the (p_1, q_1) plane; this gives

$$J = \frac{4\pi}{(ab)^{\frac{1}{2}}\,|x|}\lim_{\epsilon\to 0}\int_0^{\delta}\frac{r\,\mathrm{d}r}{r^2+t-t_0-\mathrm{i}\epsilon},$$

or

$$J = \frac{2\pi}{(ab)^{\frac{1}{2}}\,|x|}\lim_{\epsilon\to 0}\ln\left(\frac{\delta^2+t-t_0-\mathrm{i}\epsilon}{t-t_0-\mathrm{i}\epsilon}\right). \tag{2.2.13}$$

It now follows immediately from (2.2.13) that, when $t-t_0$ is small,

$$J \sim \frac{-2\pi}{(ab)^{\frac{1}{2}}\,|x|}\{\ln|t-t_0|-\mathrm{i}\pi H(t_0-t)\}. \tag{2.2.14}$$

Thus, from (2.2.1), (2.2.6) and (2.2.14),

$$\partial_t^{(n+2)}u(t,x) \sim \frac{-4\pi}{(ab)^{\frac{1}{2}}\,|x|}\frac{1}{|\nabla\omega|}\frac{\partial\omega}{\partial\xi_3}H\left(-\frac{\partial\omega}{\partial\xi_3}\right)$$
$$\times\,\mathrm{Re}\{B^N(1,\xi_\alpha^0)\,\tilde{T}(1,\xi_\alpha^0)\,(-\mathrm{i})^{n+1}\,(\ln|t-t_0|-\mathrm{i}\pi H(t_0-t))\}\quad\text{as}\quad t-t_0\to 0. \tag{2.2.15}$$

(ii) $a < 0$, $b < 0$.

This case may be dealt with similarly, by defining this time

$$p_1 = (-\tfrac{1}{2}a\,|x|)^{\frac{1}{2}}p, \quad q_1 = (-\tfrac{1}{2}b\,|x|)^{\frac{1}{2}}q. \tag{2.2.16}$$

Equation (2.2.11) is then replaced by

$$J = \frac{-2}{(ab)^{\frac{1}{2}}\,|x|}\lim_{\epsilon\to 0}\iint\frac{\mathrm{d}p_1\,\mathrm{d}q_1}{p_1^2+q_1^2+t_0-t+\mathrm{i}\epsilon}, \tag{2.2.17}$$

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in which $(ab)^{\frac{1}{2}}$ is taken positive. Similar manipulations now show that

$$J \sim \frac{2\pi}{(ab)^{\frac{1}{2}}|x|}\{\ln|t-t_0|+\mathrm{i}\pi H(t-t_0)\}, \tag{2.2.18}$$

so that

$$\partial_t^{(n+1)}u(t,x) \sim \frac{4\pi}{(ab)^{\frac{1}{2}}|x|}\frac{1}{|\nabla\omega|}\frac{\partial\omega}{\partial\xi_3}H\left(-\frac{\partial\omega}{\partial\xi_3}\right)$$
$$\times\operatorname{Re}\{B^N(1,\xi_\alpha^0)\,\tilde{T}(1,\xi_\alpha^0)\,(-\mathrm{i})^{n+1}(\ln|t-t_0|+\mathrm{i}\pi H(t-t_0))\}\quad\text{as}\quad t-t_0\to 0. \tag{2.2.19}$$

(iii) $ab<0$.

It is clear by symmetry that the same result will be obtained if either $a<0$ or $b<0$; for the calculation below, we take $a>0$ and $b<0$. Upon setting

$$p_1=(\tfrac{1}{2}a|x|)^{\frac{1}{2}}p+(-\tfrac{1}{2}b|x|)^{\frac{1}{2}}q,\quad q_1=(\tfrac{1}{2}a|x|)^{\frac{1}{2}}p-(-\tfrac{1}{2}b|x|)^{\frac{1}{2}}q, \tag{2.2.20}$$

the integral J defined by (2.2.9) becomes

$$J=\frac{1}{|ab|^{\frac{1}{2}}|x|}\lim_{\epsilon\to 0}\iint\frac{\mathrm{d}p_1\,\mathrm{d}q_1}{p_1q_1+t-t_0-\mathrm{i}\epsilon}. \tag{2.2.21}$$

It is convenient now to let $\mathcal{N}(\xi^0)$ be the square

$$|p_1|<\delta,\quad |q_1|<\delta \tag{2.2.22}$$

and also to calculate first

$$\partial_tJ=-\frac{1}{|ab|^{\frac{1}{2}}|x|}\lim_{\epsilon\to 0}\iint\frac{\mathrm{d}p_1\,\mathrm{d}q_1}{(p_1q_1+t-t_0-\mathrm{i}\epsilon)^2}. \tag{2.2.23}$$

Performing the integrations now gives

$$\partial_tJ=-\frac{2}{|ab|^{\frac{1}{2}}|x|}\lim_{\epsilon\to 0}\left\{\frac{1}{t-t_0-\mathrm{i}\epsilon}\ln\left(\frac{-\delta^2+t-t_0-\mathrm{i}\epsilon}{\delta^2+t-t_0-\mathrm{i}\epsilon}\right)\right\}.$$

Hence,

$$\partial_tJ\sim\frac{2\pi\mathrm{i}}{|ab|^{\frac{1}{2}}|x|}\frac{1}{t-t_0-0\mathrm{i}}\quad\text{as}\quad t-t_0\to 0 \tag{2.2.24}$$

and

$$J\sim\frac{2\pi\mathrm{i}}{|ab|^{\frac{1}{2}}|x|}\{\ln|t-t_0|-\mathrm{i}\pi H(t_0-t)\}. \tag{2.2.25}$$

Thus,

$$\partial_t^{(n+2)}u(t,x)\sim\frac{-4\pi}{|ab|^{\frac{1}{2}}|x|}\frac{1}{|\nabla\omega|}\frac{\partial\omega}{\partial\xi_3}H\left(-\frac{\partial\omega}{\partial\xi_3}\right)$$
$$\times\operatorname{Re}\{B^N(1,\xi_\alpha^0)\,\tilde{T}(1,\xi_\alpha^0)\,(-\mathrm{i})^{n+2}(\ln|t-t_0|-\mathrm{i}\pi H(t_0-t))\}\quad\text{as}\quad t-t_0\to 0. \tag{2.2.26}$$

For the subsonic loading which is being considered, a Rayleigh wave arrival will appear at points x on the surface $x_3=0$ of the half-space, so long as a Rayleigh wave exists. A discussion of the existence of Rayleigh waves has recently been given by Burridge (1970, 1971) and attention will be confined here to finding the Rayleigh wave arrival when it exists. It is convenient now to employ the representation (2.1.31), with $x_3=0$. The arrival is associated with poles on the real axis of the function

$$B(\omega,\eta_\alpha)=\sum_{N=4}^{6}B^N(\omega,\eta_\alpha). \tag{2.2.27}$$

Equation (2.1.21) shows that these will arise from zeros of the function $|\psi(\omega,\eta)|$, where the matrix $\psi(\omega,\eta)$ is defined as

$$\psi(\omega,\eta)=\sum_{M=4}^{6}C(\xi^M(\omega,\eta))\operatorname{adj}K(\omega,\xi^M(\omega,\eta)). \tag{2.2.28}$$

It was shown by Burridge (1970, 1971) that $|\psi(\omega,\eta)|$ may be zero for some real $\omega = \Omega^0(\eta)$ with $|\Omega^0| < c(\eta_1,\eta_2,0)$ for all wave speeds c. By letting $x_3 \to 0$ in (2.1.31) and using the definition (2.1.21) of B^N, it is obtained that

$$\partial_t^{(n+2)} u(t,x) = \frac{(2\pi)^{-\frac{1}{2}}(-\mathrm{i})^{n+1}}{t} \sum_{N=4}^{6} \oint_{|\eta|=1} \mathrm{d}s\, \Omega^{n+2} \operatorname{adj} K(\Omega, \xi^N(\Omega,\eta)) \frac{\operatorname{adj}\psi(\Omega,\eta)}{|\psi(\Omega,\eta)|} \tilde{T}(\Omega_1,\eta_\alpha), \quad (2.2.29)$$

when $x_3 = 0$, where

$$\Omega = -\eta_\alpha x_\alpha/t + 0\mathrm{i}. \quad (2.2.30)$$

Again, by the argument employed at the end of § 2.1, the right side of (2.2.29) can be replaced by twice the real part of the same expression integrated over any half of the circle $|\eta| = 1$, say the half for which $\eta_\alpha x_\alpha < 0$. The integral in (2.2.29) is singular when $-\eta_\alpha x_\alpha/t = \Omega^0(\eta)$. The effect of this singularity may be investigated by projecting the integral over the semicircle $|\eta| = 1$, $\eta_\alpha x_\alpha < 0$ onto the corresponding part C_- of the 'Rayleigh slowness curve' C, which is defined by the equation

$$\Omega^0(\zeta) = 1, \quad (2.2.31)$$

where ζ is the vector (ζ_1, ζ_2). This gives, upon taking into account the homogeneity of the integrand,

$$\partial_t^{(n+2)} u(t,x) = \frac{2(2\pi)^{-\frac{1}{2}}}{t} \operatorname{Re}\left\{(-\mathrm{i})^{n+1} \sum_{N=4}^{6} \int_{C_-} \frac{\mathrm{d}s}{|\nabla\Omega^0(\zeta)|} \operatorname{adj} K(\Omega, \xi^N(\Omega,\zeta)) \frac{\operatorname{adj}\psi(\Omega,\zeta)}{|\psi(\Omega,\zeta)|} \tilde{T}(\Omega,\zeta)\right\}, \quad (2.2.32)$$

in which

$$\Omega = -\zeta_\alpha x_\alpha/t + 0\mathrm{i}. \quad (2.2.33)$$

The Rayleigh wave arrival occurs at the time $t = t_0$ when $|\psi(\Omega,\zeta)|$ first becomes zero, that is, when the line

$$t + \zeta_\alpha x_\alpha = 0 \quad (2.2.34)$$

becomes tangent to C. To investigate the arrival, it suffices to study the behaviour of the integral

$$I_1 = \frac{1}{t} \int_{\mathscr{N}(\zeta^0)} \frac{\mathrm{d}s}{|\nabla\Omega^0(\zeta)|} \operatorname{adj} K(\Omega, \xi^N(\Omega,\zeta)) \frac{\operatorname{adj}\psi(\Omega,\zeta)}{|\psi(\Omega,\zeta)|} \tilde{T}(\Omega,\zeta), \quad (2.2.35)$$

for small $(t-t_0)$, where $\mathscr{N}(\zeta^0)$ is a small arc of C_- containing the point of tangency ζ^0. On this arc, the equation of C_- may be approximated by

$$(\zeta_\alpha^0 - \zeta_\alpha)\, x_\alpha = \tfrac{1}{2}a|x|\, s^2, \quad (2.2.36)$$

where a is the curvature of C at the point ζ^0, so long as $a \neq 0$. Also

$$|\psi(\Omega,\zeta)| \sim (-\zeta_\alpha x_\alpha/t + 0\mathrm{i} - 1)\, \partial_\Omega |\psi(1,\zeta^0)|, \quad (2.2.37)$$

since $|\psi(1,\zeta^0)| \equiv 0$ on C, and the remaining terms in the integrand may be replaced by their values at $(1,\zeta^0)$ since they are continuous. The problem thus reduces to the evaluation of the integral

$$J_1 = \int \frac{\mathrm{d}s}{t_0 - t + 0\mathrm{i} + \frac{1}{2}a|x|\, s^2}, \quad (2.2.38)$$

in which the denominator of the integrand is the asymptotic form of $t|\psi(\Omega,\zeta)|$ for ζ close to ζ^0 and t close to t_0. The integral J_1 is readily evaluated, to show that, as $t - t_0 \to 0$,

$$J_1 \sim \left(\frac{2}{a|x|}\right)^{\frac{1}{2}} \frac{\pi}{(t_0 - t + 0\mathrm{i})^{\frac{1}{2}}} \quad \text{if} \quad a > 0 \quad (2.2.39)$$

and

$$J_1 \sim -\left(\frac{2}{-a|x|}\right)^{\frac{1}{2}} \frac{\pi}{(t - t_0 - 0\mathrm{i})^{\frac{1}{2}}} \quad \text{if} \quad a < 0. \quad (2.2.40)$$

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Hence, correspondingly, if $a > 0$,

$$\partial_t^{(n+2)} u(t,x) \sim (\pi a|x|)^{-\frac{1}{2}} \operatorname{Re}\left\{\frac{(-\mathrm{i})^{n+1}}{|\nabla \Omega^0(\zeta^0)|\,\partial_\Omega|\psi(1,\zeta^0)|}\sum_{N=4}^{6} \operatorname{adj} K(1,\xi^N(1,\zeta^0)) \times \operatorname{adj}\psi(1,\zeta^0)\,\tilde{T}(1,\zeta^0_\alpha)\,(t_0-t+0\mathrm{i})^{-\frac{1}{2}}\right\}, \quad (2.2.41)$$

while if $a < 0$,

$$\partial_t^{(n+2)} u(t,x) \sim -(\pi|a|\,|x|)^{-\frac{1}{2}} \operatorname{Re}\left\{\frac{(-\mathrm{i})^{n+1}}{|\nabla \Omega^0(\zeta^0)|\,\partial_\Omega|\psi(1,\zeta^0)|}\sum_{N=4}^{6} \operatorname{adj} K(1,\xi^N(1,\zeta^0)) \times \operatorname{adj}\psi(1,\zeta^0)\,\tilde{T}(1,\zeta^0_\alpha)\,(t-t_0-0\mathrm{i})^{-\frac{1}{2}}\right\}. \quad (2.2.42)$$

2.3. *The two-dimensional problem*

Consider now the particular case of (2.1.1) in which the traction vector $T(t,x_\alpha)$ is constant along each line

$$\nu_\alpha x_\alpha = pt, \quad (2.3.1)$$

where $\nu = (\nu_1, \nu_2)$ is a constant vector in the (x_1, x_2) plane. Thus,

$$T(t,x_\alpha) = t^n T(p) \quad (t > 0). \quad (2.3.2)$$

Correspondingly, $u(t,x)$ depends only upon (t, pt, x_3) and is homogeneous of degree $n+1$ in these variables. The method of § 2.1 could be applied from first principles to find $u(t,x)$ but, since the formulae of § 2.1 are already available, an alternative procedure is to start from equation (2.1.31). It is necessary only to find the form that the Fourier transform $\tilde{T}(\Omega,\eta_\alpha)$ takes for the loading (2.3.2); we have

$$\tilde{T}(\Omega,\eta_\alpha) = (2\pi)^{-\frac{3}{2}}\int_0^\infty t^n\,\mathrm{d}t\iint \mathrm{d}x_1\,\mathrm{d}x_2\,T(p)\exp\{\mathrm{i}(\Omega t+\eta_\alpha x_\alpha)\} \quad (2.3.3)$$

or, upon introducing the coordinate q as the projection of x/t onto a unit vector λ in the (x_1, x_2) plane which is perpendicular to ν,

$$\tilde{T}(\Omega,\eta_\alpha) = (2\pi)^{-\frac{3}{2}}\int_0^\infty t^{n+2}\,\mathrm{d}t\iint \mathrm{d}p\,\mathrm{d}q\,T(p)\exp\{\mathrm{i}t(\Omega+\eta_\alpha\nu_\alpha p+\eta_\alpha\lambda_\alpha q)\}. \quad (2.3.4)$$

The integration with respect to q may be performed to yield a factor $2\pi t^{-1}\delta(\lambda_\alpha\eta_\alpha)$, after which integration with respect to t gives

$$\tilde{T}(\Omega,\eta_\alpha) = (-\mathrm{i})^n(2\pi)^{-\frac{1}{2}}\,\partial_\Omega^{(n+1)}\left\{\delta_1(\lambda_\alpha\eta_\alpha)\int\frac{\mathrm{d}p\,T(p)}{p+\Omega}-\delta_2(\lambda_\alpha\eta_\alpha)\int\frac{\mathrm{d}p\,T(p)}{p-\Omega}\right\}, \quad (2.3.5)$$

where

$$\left.\begin{aligned}\delta_1(\lambda_\alpha\eta_\alpha) &= \delta(\lambda_\alpha\eta_\alpha)\,H(\nu_\alpha\eta_\alpha),\\ \delta_2(\lambda_\alpha\eta_\alpha) &= \delta(\lambda_\alpha\eta_\alpha)\,H(-\nu_\alpha\eta_\alpha).\end{aligned}\right\} \quad (2.3.6)$$

The presence of the delta functions in (2.3.5) immediately reduces the integral around the contour $|\eta| = 1$, when (2.3.5) is substituted into (2.1.31). There are contributions only from the points $\eta = \nu$ and $\eta = -\nu$ which, furthermore, can be shown to be complex conjugates, by the reasoning following equation (2.1.36). Therefore,

$$\partial_t^{(n+2)} u(t,x) = 2(2\pi)^{-\frac{3}{2}}(-1)^n\sum_{N=4}^{6}\operatorname{Re}\left\{\frac{(\Omega^N)^{n+2}B^N(\Omega^N,\nu_\alpha)}{t+\xi^N_{3,\Omega}(\Omega^N,\nu)\,x_3}\,\partial_{\Omega^N}^{(n+1)}\int\frac{\mathrm{d}p\,T(p)}{p+\Omega^N}\right\}H(t-t_N), \quad (2.3.7)$$

where now $\Omega^N = \Omega^N(\nu)$. Equation (2.3.7) can be expressed in a form similar to equation (2.1.35) by defining

$$G^{(n+1)}(z,\nu) = \frac{1}{2\pi i}\partial_z^{(n+1)}\int\frac{dp\,T(p)}{p-z}. \tag{2.3.8}$$

Substitution of this expression into (2.3.7) gives

$$\partial_t^{(n+2)}u(t,x) = 2(2\pi)^{-\frac{1}{2}}\sum_{N=4}^{6}\mathrm{Im}\left\{\frac{(\Omega^N)^{n+2}B^N(\Omega^N,\nu_\alpha)}{t+\xi^N_{3,\Omega}(\Omega^N,\nu)\,x_3}G^{(n+1)}(-\Omega^N,\nu)\right\}H(t-t_N), \tag{2.3.9}$$

where the symbol Im denotes 'the imaginary part of'.

Equation (2.3.9), like (2.1.35), can be integrated once with respect to time by transforming the variable of integration to Ω^N; this yields

$$\partial_t^{(n+1)}u(t,x) = 2(2\pi)^{-\frac{1}{2}}\sum_{N=4}^{6}\mathrm{Im}\left\{\int_{\Omega^N(t,x,\nu)}^{\infty}d\Omega\,\Omega^{n+1}B^N(\Omega,\nu_\alpha)\,G^{(n+1)}(-\Omega,\nu)\right\}H(t-t_N). \tag{2.3.10}$$

2.4. *The arrivals for the two-dimensional problem*

The singularities in the field $\partial_t^{(n+2)}u(t,x)$ represented by (2.3.9) are easier to find than their three-dimensional counterparts, since they will arise just from the singularities of the algebraic functions $B^N(\Omega^N,x_\alpha)/(t+\xi^N_{3,\Omega}(\Omega^N,\nu)\,x_3)$, at least if the loading $T(t,x_\alpha)$ is subsonic in the sense that $T(t,x_\alpha) = 0$ if $|\nu_\alpha x_\alpha| > Vt$, for some V smaller than all of the speeds of plane waves propagating in the direction of ν; this will be assumed in what follows.

First, the body-wave arrivals are produced by the zeros of the functions

$$\chi^N(t,x,\nu) = t+\xi^N_{3,\Omega}(\Omega^N,\nu)\,x_3. \tag{2.4.1}$$

The time $t = t_0$ at which $\chi^N(t,x,\nu)$ is zero is closely related to the geometry of the intersection of the slowness surface (2.2.3) with the plane containing both ν and the x_3-axis, since Ω^N in (2.4.1) satisfies the equation

$$\Omega^N t+\nu_\alpha x_\alpha+\xi_3^N(\Omega^N,\nu)\,x_3 = 0, \tag{2.4.2}$$

while the definition of ξ_3^N implies that

$$|K(\Omega^N,\nu_\alpha,\xi_3^N)| = 0. \tag{2.4.3}$$

Thus, upon defining

$$s = 1/\Omega^N, \quad \xi_3 = s\xi_3^N, \tag{2.4.4}$$

it follows from (2.4.2) and (2.4.3) that

$$t+s\nu_\alpha x_\alpha+\xi_3 x_3 = 0 \tag{2.4.5}$$

and

$$\omega(s\nu_\alpha,\xi_\alpha) = 1, \tag{2.4.6}$$

where $\omega(\xi) = 1$ defines one sheet of the slowness surface (2.4.3). Equations (2.4.5) and (2.4.6) have a pair of solutions (s,ξ_3) which coalesce at the time $t = t_0$ at which the curve (2.4.6) is tangent to the line (2.4.5) in the (s,ξ_3) plane. That $\chi^N(t,x,\nu) = 0$ when $t = t_0$ follows since the vanishing of χ^N is just the condition that equation (2.4.2) should have a double root, $\Omega^N = \Omega^0$, say. This can be verified explicitly by noting that equation (2.4.6) implies

$$\xi_3 = \xi_3^N(1,s\nu_\alpha), \tag{2.4.7}$$

which is tangent to (2.4.5) when

$$\nu_\alpha x_\alpha+\xi^N_{3,s}x_3 = 0. \tag{2.4.8}$$

But ξ_3^N is homogeneous of degree 1 in (Ω, s) and so

$$s\xi^N_{3,s}(1, s\nu_\alpha) + \xi^N_{3,\Omega}(1, s\nu_\alpha) + \xi_3^N(1, s\nu_\alpha). \tag{2.4.9}$$

Therefore, (2.4.8) is equivalent to

$$s\nu_\alpha x_\alpha + \xi_3^N(1, s\nu_\alpha)\, x_3 - \xi^N_{3,\Omega}(1, s\nu_\alpha)\, x_3 = 0. \tag{2.4.10}$$

Since equation (2.4.5) is satisfied with this value of s when $t = t_0$ and $\xi^N_{3\ \Omega}$ is homogeneous of degree zero, equation (2.4.10) states that

$$\chi^N(t_0, x, \nu) = 0.$$

The asymptotic form of $\chi^N(t, x, \nu)$ for t close to t_0 may now be found by a straightforward perturbation expansion. A similar exercise has recently been performed by Payton (1972), who investigated in great detail the arrivals produced by a line source in an infinite isotropic medium which had been uniformly prestrained. Here, as in § 2.2, attention will be restricted to the most common case in which the curvature of (2.4.6) is finite and non-zero at the point of tangency (s^0, ξ_3^0) with (2.4.5). Let $\Omega^N = \Omega^0$ when $t = t_0$. Then, for t close to t_0,

$$\chi^N(t, x, \nu) \sim t - t_0 + (\Omega^N - \Omega^0)\, \xi^N_{3,\Omega\Omega}(\Omega^0, \nu)\, x_3. \tag{2.4.11}$$

But by expanding (2.4.2),

$$(\Omega^N - \Omega^0)\, t_0 + \Omega^0(t - t_0) + (\Omega^N - \Omega^0)\, \xi^N_{3,\Omega}(\Omega^0, \nu)\, x_3 + \tfrac{1}{2}(\Omega^N - \Omega^0)^2\, \xi^N_{3.\Omega\Omega}(\Omega^0, \nu)\, x_3 \sim 0, \tag{2.4.12}$$

so that

$$(\Omega^N - \Omega^0) \sim \left(\frac{2\Omega^0(t_0 - t)}{\xi^N_{3,\Omega\Omega}(\Omega^0, \nu)\, x_3}\right)^{\frac{1}{2}}, \tag{2.4.13}$$

since $\chi^N(t_0, x, \nu) = 0$. The branch of the square root in (2.4.13) is chosen so that $\Omega^N - \Omega^0$ has non-negative imaginary part. Hence

$$\left.\begin{aligned} \chi^N(t, x, \nu) &\sim \{2\Omega^0 \xi^N_{3,\Omega\Omega}(\Omega^0, \nu)\, x_3(t_0 - t)\}^{\frac{1}{2}} \qquad (\xi^N_{3,\Omega\Omega} > 0), \\ &\sim -\{2\Omega^0 \xi^N_{3,\Omega\Omega}(\Omega^0, \nu)\, x_3(t_0 - t)\}^{\frac{1}{2}} \quad (\xi^N_{3,\Omega\Omega} < 0). \end{aligned}\right\} \tag{2.4.14}$$

The term $\xi^N_{3,\Omega\Omega}(\Omega^0, \nu)$ can, of course, be related to the curvature of (2.4.6) at (s^0, ξ_3^0); a more compact expression is obtained, however, by noting that

$$\Omega^0 \xi^N_{3,\Omega\Omega}(\Omega^0, \nu) = \xi^N_{3,\Omega\Omega}(1, s^0\nu), \tag{2.4.15}$$

since this function is homogeneous of degree zero. Hence, for t close to t_0,

$$\begin{aligned} \partial_t^{(n+2)} u(t, x) \sim 2(2\pi)^{-\frac{1}{2}}\, \mathrm{Im}\,\{(\Omega^0)^{n+2} B^N(\Omega^0, \nu_\alpha)\, G^{(n+1)}(-\Omega^0, \nu) \\ \times [2\xi^N_{3,\Omega\Omega}(1, s^0\nu)\, x_3\, (t_0 - t)]^{-\frac{1}{2}}\}\, \mathrm{sgn}\, [\xi^N_{3,\Omega\Omega}(1, s^0\nu)], \end{aligned} \tag{2.4.16}$$

where

$$\Omega^0 = 1/s^0 \tag{2.4.17}$$

and the curve (2.4.6) touches the line (2.4.5) at $(s^0, \xi_3^N(1, s^0\nu))$ when $t = t_0$.

There is a Rayleigh arrival on the surface $x_3 = 0$ if the function $B(\Omega, \nu_\alpha)$ which is defined by equation (2.2.27) has a pole on the real axis. As discussed in § 2.2, $B(\Omega, \nu_\alpha)$ has a pole at $\Omega = \Omega^0(\nu)$ if

$$|\psi(\Omega^0, \nu)| = 0, \tag{2.14.18}$$

where $\psi(\omega, \eta)$ is defined by equation (2.2.28); $\Omega^0(\nu)$ if it exists, is the speed with which a Rayleigh wave will travel in the direction of ν. Then, if t_0 is defined by

$$t_0 = |\nu_\alpha x_\alpha| / \Omega^0, \tag{2.4.19}$$

it follows that
$$t|\psi(-\nu_\alpha x_\alpha/t+0\mathrm{i},\nu)| \sim (\Omega^0(t_0-t)+0\mathrm{i})\,\partial_\Omega|\psi(\Omega^0,\nu)|, \tag{2.4.20}$$
when t is close to t_0, if $\nu_\alpha x_\alpha < 0$. Hence,
$$\partial_t^{(n+2)}u(t,x) \sim -2(2\pi)^{-\frac{1}{2}}\,\mathrm{Re}\left\{(\Omega^0)^{n+2}\sum_{N=4}^{6}\operatorname{adj}K(\Omega^0,\xi^N(\Omega^0,\nu))\operatorname{adj}\psi(\Omega^0,\nu) + \frac{G^{(n+1)}(-\Omega^0,\nu)}{[\Omega^0(t_0-t)+0\mathrm{i}]\,\partial_\Omega|\psi(\Omega^0,\nu)|}\right\}. \tag{2.4.21}$$
If $\nu_\alpha x_\alpha > 0$, the corresponding result is (2.4.21) with t and t_0 interchanged.

2.5. *Lamb's problem*

Lamb's problem in three dimensions is to find the displacements generated by a point impulse applied at the origin at time $t = 0$. Thus
$$T(t,x_\alpha) = T_0\,\delta(t)\,\delta(x_1)\,\delta(x_2), \tag{2.5.1}$$
where T_0 is a constant vector. The displacement $u(t,x)$ which is produced by (2.5.1) is most conveniently expressed in the form
$$u(t,x) = U(t,x)\,T_0, \tag{2.5.2}$$
where the matrix $U(t,x)$ satisfies the differential equation (2.1.3), the initial conditions (2.1.4) and the boundary conditions (2.1.5) but with $T(t,x_\alpha)$ replaced by $I\delta(t)\,\delta(x_1)\,\delta(x_2)$, I being the identity. The problem does not fit immediately into the scheme of § 2.1 since it was implicitly assumed there that $n \geqslant -2$. However, Duhamel's principle ensures that
$$U(t,x) = \partial_t\,U_1(t,x), \tag{2.5.3}$$
where the matrix $U_1(t,x)$ satisfies (2.1.5) with $T(t,x_\alpha)$ replaced by $IH(t)\,\delta(x_1)\,\delta(x_2)$. The matrix $U_1(t,x)$ can be found immediately from equation (2.1.31), with $n = -2$ and
$$\tilde{T}(\Omega,\eta_\alpha) = (2\pi)^{-\frac{3}{2}}I\int_0^\infty \mathrm{d}t\iint \mathrm{d}x_1\,\mathrm{d}x_2\exp\{\mathrm{i}(\Omega t+\eta_\alpha x_\alpha)\}\,\delta(x_1)\,\delta(x_2), \tag{2.5.4}$$
that is,
$$\tilde{T}(\Omega,\eta_\alpha) = (2\pi)^{-\frac{3}{2}}I\mathrm{i}/\Omega. \tag{2.5.5}$$
Hence
$$U(t,x) = (2\pi)^{-\frac{1}{2}}\mathrm{i}\partial_t\sum_{N=6}^{6}\oint_{|\eta|=1}\mathrm{d}s\,\frac{B^N(\Omega^N,\eta_\alpha)\,H(t-t_N)}{\Omega^N[t+\xi^n_{3,\Omega}(\Omega^N,\eta)\,x_3]}. \tag{2.5.6}$$
Equation (2.5.6) is easily shown to be equivalent to the result of Burridge (1971), which was expressed in terms of a 'slowness' variable $s^N = 1/\Omega^N$ and was obtained by an application of Cagniard's technique.

The 'arrivals' associated with (2.5.6) were not discussed by Burridge. They are obtainable directly from the results of § 2.2. For example, if $a > 0$ and $b > 0$ in the notation of § 2.2, the body-wave arrival is given as
$$U(t,x) \sim \frac{2(2\pi)^{-\frac{1}{2}}}{(ab)^{\frac{1}{2}}|x|}\,\frac{1}{|\nabla\omega|}\,\frac{\partial\omega}{\partial\xi_3}H\left(-\frac{\partial\omega}{\partial\xi_3}\right)\mathrm{Re}\left\{\frac{B^N(1,\xi^0_\alpha)}{t-t_0-0\mathrm{i}}\right\} \tag{2.5.7}$$
as $t-t_0\to 0$. The other cases, $a < 0$ and $b < 0$, and $ab < 0$, could be given similarly. The Rayleigh arrival can likewise be obtained, from (2.2.41) if $a > 0$ or from (2.2.42) if $a < 0$. If $a > 0$, for example,
$$U(t,x) \sim -\frac{1}{8\pi^2}\left(\frac{2}{a|x|}\right)^{\frac{1}{2}}\mathrm{Re}\left\{\sum_{N=4}^{6}\frac{\operatorname{adj}K(1,\xi^N(1,\zeta^0))\operatorname{adj}(1,\zeta^0)}{|\nabla\Omega^0(\zeta^0)|\,\partial_\Omega|\psi(1,\zeta^0)|}(t_0-t+0\mathrm{i})^{-\frac{3}{2}}\right\}, \tag{2.5.8}$$
as $t-t_0\to 0$, for $x_3 = 0$.

The two-dimensional form of Lamb's problem is to find $u(t, x)$ for the loading

$$T(t, x_\alpha) = T_0\,\delta(t)\,\delta(\nu_\alpha x_\alpha), \tag{2.5.9}$$

where ν is a unit vector in the (x_1, x_2) plane. Again, $u(t, x)$ has the form (2.5.2), where now $U(t, x)$ satisfies (2.1.3) and (2.1.4), and (2.1.5) with $T(t, x_\alpha) = I\,\delta(t)\,\delta(\nu_\alpha x_\alpha)$. This loading is homogeneous of degree -2 but the result (2.3.9) is not directly applicable because it is strictly valid only for $n \geqslant -1$. However, $U(t, x)$ is given by (2.1.31) with $n = -2$ and

$$\tilde{T}(\Omega, \eta_\alpha) = (2\pi)^{-\frac{3}{2}}\,\delta(\lambda_\alpha \eta_\alpha), \tag{2.5.10}$$

where λ is a unit vector in the (x_1, x_2) plane which is perpendicular to ν. As in the derivation of (2.3.9), the contributions to the integral (2.1.31) from $\eta = \nu$ and $\eta = -\nu$ are complex conjugates, and so

$$U(t, x) = 2(2\pi)^{-\frac{1}{2}} \sum_{N=4}^{6} \operatorname{Re}\left\{\frac{B^N(\Omega^N, \nu_\alpha)}{t + \xi^N_{3,\Omega}(\Omega^N, \nu)\,x_3}\right\} H(t - t_N), \tag{2.5.11}$$

which again agrees with Burridge (1971).

It may be noted that equation (2.5.11) has the form (2.3.9) with $n = -2$, if $G^{(n+1)}(-\Omega^N, \nu)$ is replaced by $-\mathrm{i}$. The 'arrivals' of the field (2.5.11) are therefore obtainable directly from the results of § 2.4, if this replacement is made. For the body-wave arrivals, from (2.4.16),

$$U(t, x) \sim -2(2\pi)^{-\frac{1}{2}} \operatorname{Re}\{B^N(\Omega^0, \nu_\alpha)\,[2\xi^N_{3,\Omega\Omega}(1, s^0\nu)\,x_3(t_0 - t)]^{-\frac{1}{2}}\} \operatorname{sgn}[\xi^N_{3,\Omega\Omega}(1, s^0\nu)], \tag{2.5.12}$$

as $t - t_0 \to 0$, while for the Rayleigh arrival, if $\nu_\alpha x_\alpha < 0$,

$$U(t, x) \sim -2(2\pi)^{-\frac{1}{2}} \operatorname{Im}\left\{\sum_{N=4}^{6} \frac{\operatorname{adj} K(\Omega^0, \xi^N(\Omega^0, \nu))\,\operatorname{adj}\psi(\Omega^0, \nu)}{[\Omega^0(t_0 - t) + 0\mathrm{i}]\,\partial_\Omega|\psi(\Omega^0, \nu)|}\right\}, \tag{2.5.13}$$

as $t - t_0 \to 0$, for $x_3 = 0$, from (2.4.21). If $\nu_\alpha x_\alpha > 0$, the Rayleigh arrival is given by (2.5.13) with t and t_0 interchanged.

2.6. *Isotropic half-space*

The results of the preceding sections will be specialized for an isotropic half-space both for detailed illustration and for later use. If the half-space has Lamé moduli λ, μ, the matrix $K(\omega, \xi)$ has components

$$K_{ik}(\omega, \xi) = (\mu\xi_j\xi_j - \rho\omega^2)\,\delta_{ik} + (\lambda + \mu)\,\xi_i\xi_k \tag{2.6.1}$$

and a simple calculation shows that equation (2.1.12) has roots

$$\left.\begin{aligned} \xi_3^{\pm\alpha} &= \pm(\omega^2/\alpha^2 - |\xi|^2)^{\frac{1}{2}} \quad \text{(once)}, \\ \xi_3^{\pm\beta} &= \pm(\omega^2/\beta^2 - |\xi|^2)^{\frac{1}{2}} \quad \text{(twice)}, \end{aligned}\right\} \tag{2.6.2}$$

where α and β, the speeds of dilatational and shear waves, are given as

$$\alpha^2 = (\lambda + 2\mu)/\rho, \quad \beta^2 = \mu/\rho \tag{2.6.3}$$

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and, as previously, $|\xi|^2 = \xi_1^2+\xi_2^2$. In view of the degeneracy of the matrix $K(\omega, \xi)$, the reasoning of § 2.1 needs slight modification. First, it may be noted that $K(\omega, \xi)$ has eigenvectors

$$U^{\pm\alpha} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3^{\pm\alpha} \end{bmatrix}, \quad U_1^{\pm\beta} = \begin{bmatrix} \xi_1 \xi_3^{\pm\beta}/|\xi| \\ \xi_2 \xi_3^{\pm\beta}/|\xi| \\ -|\xi| \end{bmatrix}, \quad U_2^{\pm\beta} = \begin{bmatrix} -\xi_2 \\ \xi_1 \\ 0 \end{bmatrix}, \tag{2.6.4}$$

the former pair corresponding to the eigenvalues $\xi_3^{\pm\alpha}$ and the latter two pairs to the double eigenvalues $\xi_3^{\pm\beta}$. In terms of these, the general solution of equation (2.1.9) which is bounded as $x_3 \to \infty$ with $\mathrm{Im}\,(\omega) > 0$ is

$$\tilde{u}(\omega, \xi_1, \xi_2, x_3) = U^{-\alpha} b^\alpha \exp\{-\mathrm{i}\xi_3^{-\alpha} x_3\} + (U_1^{-\beta} b_1^\beta + U_2^{-\beta} b_2^\beta) \exp\{-\mathrm{i}\xi_3^{-\beta} x_3\}, \tag{2.6.5}$$

where b^α, b_1^β and b_2^β are functions of (ω, ξ_1, ξ_2), if the radicals in (2.6.2) are defined by cutting the complex ω-plane between $\pm\alpha|\xi|$ or $\pm\beta|\xi|$ and the branches are chosen so that $\mathrm{Im}\,(\xi_3^{-\alpha})$, $\mathrm{Im}\,(\xi_3^{-\beta}) < 0$ when $\mathrm{Im}\,(\omega) > 0$. Equations (2.1.17) are now replaced by

$$\begin{bmatrix} 2\mathrm{i}\xi_1\left(\dfrac{\omega^2}{\alpha^2}-|\xi|^2\right)^{\frac{1}{2}} & \dfrac{\mathrm{i}\xi_1}{|\xi|}\left(2|\xi|^2-\dfrac{\omega^2}{\beta^2}\right) & -\mathrm{i}\xi_2\left(\dfrac{\omega^2}{\beta^2}-|\xi|^2\right)^{\frac{1}{2}} \\ 2\mathrm{i}\xi_2\left(\dfrac{\omega^2}{\alpha^2}-|\xi|^2\right)^{\frac{1}{2}} & \dfrac{\mathrm{i}\xi_2}{|\xi|}\left(2|\xi|^2-\dfrac{\omega^2}{\beta^2}\right) & \mathrm{i}\xi_1\left(\dfrac{\omega^2}{\beta^2}-|\xi|^2\right)^{\frac{1}{2}} \\ \mathrm{i}\left(2|\xi|^2-\dfrac{\omega^2}{\beta^2}\right) & -2\mathrm{i}|\xi|\left(\dfrac{\omega^2}{\beta^2}-|\xi|^2\right)^{\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} b^\alpha \\ b_1^\beta \\ b_2^\beta \end{bmatrix} = \tilde{T}(\omega, \xi_\alpha)/\mu \tag{2.6.6}$$

which may be solved to give

$$\begin{bmatrix} b^\alpha \\ b_1^\beta \\ b_2^\beta \end{bmatrix} = \frac{1}{\mathrm{i}\mu|\xi|\,D(\omega,|\xi|)} \begin{bmatrix} 2\xi_1|\xi|\left(\dfrac{\omega^2}{\beta^2}-|\xi|^2\right)^{\frac{1}{2}} & 2\xi_2|\xi|\left(\dfrac{\omega^2}{\beta^2}-|\xi|^2\right)^{\frac{1}{2}} & \left(2|\xi|^2-\dfrac{\omega^2}{\beta^2}\right)|\xi| \\ \xi_1\left(2|\xi|^2-\dfrac{\omega^2}{\beta^2}\right) & \xi_2\left(2|\xi|^2-\dfrac{\omega^2}{\beta^2}\right) & -2|\xi|^2\left(\dfrac{\omega^2}{\alpha^2}-|\xi|^2\right)^{\frac{1}{2}} \\ \dfrac{-\xi_2 D(\omega,|\xi|)}{|\xi|\left(\dfrac{\omega^2}{\beta^2}-|\xi|^2\right)^{\frac{1}{2}}} & \dfrac{\xi_1 D(\omega,|\xi|)}{|\xi|\left(\dfrac{\omega^2}{\beta^2}-|\xi|^2\right)^{\frac{1}{2}}} & 0 \end{bmatrix} \tilde{T}(\omega, \xi_\alpha), \tag{2.6.7}$$

where
$$D(\omega,|\xi|) = 4|\xi|^2\left(\frac{\omega^2}{\alpha^2}-|\xi|^2\right)^{\frac{1}{2}}\left(\frac{\omega^2}{\beta^2}-|\xi|^2\right)^{\frac{1}{2}}+\left(2|\xi|^2-\frac{\omega^2}{\beta^2}\right)^2. \tag{2.6.8}$$

The equation $D(\omega, |\xi|) = 0$ is the secular equation for Rayleigh waves, yielding values of $(\omega, |\xi|)$ for which a combination of plane waves of the type (2.6.5) can be found which leaves the boundary $x_3 = 0$ of the half-space free of traction. $\tilde{u}(\omega, \xi_1, \xi_2, x_3)$ is now obtained by substituting (2.6.7) into (2.6.5). For conciseness of expression below, note that this is expressible as

$$\tilde{u}(\omega, \xi_1, \xi_2, x_3) = (2\pi)^{\frac{3}{2}}\,\tilde{U}(\omega, \xi_1, \xi_2, x_3)\,\tilde{T}(\omega, \xi_\alpha), \tag{2.6.9}$$

where $\tilde{U}(\omega, \xi_1, \xi_2, x_3)$ is the Fourier transform of the solution $U(t, x)$ of Lamb's problem, which was discussed in § 2.5. A complete expression for $\tilde{U}(\omega, \xi_1, \xi_2, x_3)$ will not be given but, for illustration,

$$\tilde{U}_{33}(\omega, \xi_1, \xi_2, x_3) = -\frac{(\omega^2/\alpha^2-|\xi|^2)^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}\,\mathrm{i}\mu D(\omega,|\xi|)} \times\left[\left(2|\xi|^2-\frac{\omega^2}{\beta^2}\right)\exp\left\{\mathrm{i}\left(\frac{\omega^2}{\alpha^2}-|\xi|^2\right)^{\frac{1}{2}}x_3\right\}-2|\xi|^2\exp\left\{\mathrm{i}\left(\frac{\omega^2}{\beta^2}-|\xi|^2\right)^{\frac{1}{2}}x_3\right\}\right] \tag{2.6.10}$$

and the (33) component of equation (2.5.6) reduces to

$$U_{33}(t,x) = -\frac{1}{4\pi\mu}\partial_t \oint_{|\eta|=1} \mathrm{d}s\, \mathrm{Im}\left\{\frac{\left(\frac{\Omega_\alpha^2}{\alpha^2}-1\right)\left(2-\frac{\Omega_\alpha^2}{\beta^2}\right)H\left(t-\frac{x_3}{\alpha}\right)}{\mathrm{i}\Omega_\alpha D(\Omega_\alpha,1)\left[t\left(\frac{\Omega_\alpha^2}{\alpha^2}-1\right)^{\frac{1}{2}}-\frac{\Omega_\alpha x_3}{\alpha^2}\right]} - \frac{2\left(\frac{\Omega_\beta^2}{\alpha^2}-1\right)^{\frac{1}{2}}\left(\frac{\Omega_\beta^2}{\beta^2}-1\right)^{\frac{1}{2}}H\left(t-\frac{x_3}{\beta}\right)}{\mathrm{i}\Omega_\beta D(\Omega_\beta,1)\left[t\left(\frac{\Omega_\beta^2}{\beta^2}-1\right)^{\frac{1}{2}}-\frac{\Omega_\beta x_3}{\beta^2}\right]}\right\}, \tag{2.6.11}$$

where

$$\left.\begin{aligned}\Omega_\alpha t+\eta_\lambda x_\lambda-(\Omega_\alpha^2/\alpha^2-1)^{\frac{1}{2}}x_3 &= 0,\\ \Omega_\beta t+\eta_\lambda x_\lambda-(\Omega_\beta^2/\beta^2-1)^{\frac{1}{2}}x_3 &= 0.\end{aligned}\right\} \tag{2.6.12}$$

The value of the transform $\tilde{U}(\omega,\xi_1,\xi_2,x_3)$ when $x_3=0$ will be of interest in later sections. In the general case, it is equal to the function $B(\omega,\xi_\alpha)$ defined by equation (2.2.27) and this notation will be preserved here. Thus,

$$B(\omega,\xi_\alpha) = \tilde{U}(\omega,\xi_1,\xi_2,0) = \frac{1}{|\xi|}\begin{bmatrix} b+c\eta_2^2, & -c\eta_1\eta_2, & -\mathrm{i}d\eta_1\\ -c\eta_1\eta_2, & b+c\eta_1^2, & -\mathrm{i}d\eta_2\\ \mathrm{i}d\eta_1, & \mathrm{i}d\eta_2, & e\end{bmatrix}, \tag{2.6.13}$$

where

$$\left.\begin{aligned} b &= \frac{(2\pi)^{-\frac{3}{2}}}{\mathrm{i}\mu D(\Omega,1)}\frac{\Omega^2}{\beta^2}\left(\frac{\Omega^2}{\beta^2}-1\right)^{\frac{1}{2}},\\ c &= \frac{-(2\pi)^{-\frac{3}{2}}}{\mathrm{i}\mu D(\Omega,1)}\left[\frac{\Omega^2}{\beta^2}\left(\frac{\Omega^2}{\beta^2}-1\right)^{\frac{1}{2}}-D(\Omega,1)\left(\frac{\Omega^2}{\beta^2}-1\right)^{-\frac{1}{2}}\right],\\ d &= \frac{-(2\pi)^{-\frac{3}{2}}}{\mu D(\Omega,1)}\left[2-\frac{\Omega^2}{\beta^2}+2\left(\frac{\Omega^2}{\alpha^2}-1\right)^{\frac{1}{2}}\left(\frac{\Omega^2}{\beta^2}-1\right)^{\frac{1}{2}}\right],\\ e &= \frac{(2\pi)^{-\frac{3}{2}}}{\mathrm{i}\mu D(\Omega,1)}\frac{\Omega^2}{\beta^2}\left(\frac{\Omega^2}{\alpha^2}-1\right)^{\frac{1}{2}},\end{aligned}\right\} \tag{2.6.14}$$

and Ω, η are defined by equation (2.1.24). It may be noted that, when Ω is real and $|\Omega|<\beta$, the matrix $B(\omega,\xi_\alpha)$ is hermitian and that, as $\Omega\to 0$ (or, equivalently, $\rho\to 0$ so that $\alpha,\beta\to\infty$), $B(\omega,\xi_\alpha)$ is proportional to the transform of the corresponding static Green function (see, for example, Willis 1971*a*).

To discuss the body-wave arrivals in an isotropic half-space, we note that

$$\begin{aligned} \omega(\xi) &= \alpha(\xi_1^2+\xi_2^2+\xi_3^2)^{\frac{1}{2}} \quad &&\text{or}\quad \beta(\xi_1^2+\xi_2^2+\xi_3^2)^{\frac{1}{2}},\\ |\nabla\omega| &= \alpha &&\text{or}\quad \beta,\\ \xi^0 &= -\frac{1}{\alpha}\frac{x}{|x|} &&\text{or}\quad -\frac{1}{\beta}\frac{x}{|x|},\end{aligned}$$

and

$$a=b=\alpha \quad \text{or} \quad \beta.$$

Therefore, equation (2.2.15) is the relevant one or, correspondingly for $U(t,x)$, equation (2.5.7). For $U_{33}(t,x)$ the latter gives

$$U_{33}(t,x) \sim \frac{1}{2\pi\mu\alpha|x|}\frac{x_3^2}{|x|^2}\left[\frac{2(x_1^2+x_2^2)}{\alpha^2|x|^2}-\frac{1}{\beta^2}\right]\delta(\alpha t-|x|)\Big/D\left(1,\frac{(x_1^2+x_2^2)^{\frac{1}{2}}}{\alpha|x|}\right), \tag{2.6.15}$$

as $t \to |x|/\alpha$, where

$$D\left(1, \frac{(x_1^2+x_2^2)^{\frac{1}{2}}}{\alpha|x|}\right) = \frac{4(x_1^2+x_2^2)\,x_3}{\alpha^3|x|^3}\left(\frac{1}{\beta^2}-\frac{x_1^2+x_2^2}{\alpha^2|x|^2}\right)^{\frac{1}{2}} + \left(\frac{2(x_1^2+x_2^2)}{\alpha^2|x|^2}-\frac{1}{\beta^2}\right)^2, \tag{2.6.16}$$

and $$U_{33}(t,x) \sim \frac{-(x_1^2+x_2^2)\,x_3}{\pi^2\mu\beta^2|x|^3}\,\mathrm{Im}\left\{\left(\frac{1}{\alpha^2}-\frac{x_1^2+x_2^2}{\beta^2|x|^2}\right)^{\frac{1}{2}}(\beta t-|x|-0\mathrm{i})^{-1}\Big/D\left(1, \frac{(x_1^2+x_2^2)^{\frac{1}{2}}}{\beta|x|}\right)\right\} \tag{2.6.17}$$

as $t \to |x|/\beta$, where

$$D\left(1, \frac{(x_1^2+x_2^2)^{\frac{1}{2}}}{\beta|x|}\right) = \frac{4(x_1^2+x_2^2)}{|x|^3}\left(\frac{1}{\alpha^2}-\frac{x_1^2+x_2^2}{\beta^2|x|^2}\right)^{\frac{1}{2}} + \frac{1}{\beta^4}\left(\frac{2(x_1^2+x_2^2)}{|x|^2}-1\right)^2. \tag{2.6.18}$$

These expressions agree with those given by White (1965), which were obtained by an ingenious use of the Green reciprocal theorem. Equation (2.6.17) displays the well-known 'conical wave', due to the change in the analytic form of the arrival across the cone $\beta^2|x|^2-\alpha^2(x_1^2+x_2^2) = 0$. The Rayleigh arrivals can also be obtained, by noting that $D(\Omega, 1)$ takes the place of $|\psi(\Omega,\eta)|$. The Rayleigh slowness curve C becomes the circle

$$\zeta_1^2+\zeta_2^2 = 1/c_{\mathrm{R}}^2, \tag{2.6.19}$$

where the Rayleigh wave speed c_{R} satisfies the equation

$$D(c_{\mathrm{R}}, 1) = 0. \tag{2.6.20}$$

Therefore, $$\zeta^0 = -(x_1, x_2)/(c_{\mathrm{R}}|x|) \tag{2.6.21}$$

and the parameter a of equation (2.2.36) takes the value

$$a = -c_{\mathrm{R}}. \tag{2.6.22}$$

Therefore, equation (2.2.42) is relevant or, for $U(t,x)$, equation (2.5.8). Together with (2.6.14), the latter gives, for $U_{33}(t,x)$,

$$U_{33}(t,x) \sim \frac{(2\pi)^{-\frac{7}{2}}(1-c_{\mathrm{R}}^2/\alpha^2)^{\frac{1}{2}}}{(2c_{\mathrm{R}}|x|)^{\frac{1}{2}}\beta^2\mu c_{\mathrm{R}}^2 D_\omega}\,\mathrm{Re}\left\{\left(t-\frac{|x|}{c_{\mathrm{R}}}+0\mathrm{i}\right)^{-\frac{3}{2}}\right\}, \tag{2.6.23}$$

as $t \to |x|/c_{\mathrm{R}}$ and $x_3 = 0$, where

$$D_\omega = \frac{4}{c_{\mathrm{R}}^2}\left(\frac{1}{c_{\mathrm{R}}^2}-\frac{1}{\alpha^2}\right)^{-\frac{1}{2}}\left(\frac{1}{c_{\mathrm{R}}^2}-\frac{1}{\beta^2}\right)^{-\frac{1}{2}}\left(\frac{1}{\alpha^2 c_{\mathrm{R}}^2}+\frac{1}{\beta^2 c_{\mathrm{R}}^2}-\frac{2}{\alpha^2\beta^2}\right)-\frac{4}{\beta^2}\left(\frac{2}{c_{\mathrm{R}}^2}-\frac{1}{\beta^2}\right). \tag{2.6.24}$$

The (11) arrival is more interesting as it is not isotropic:

$$U_{11}(t,x) \sim \frac{(2\pi)^{-\frac{7}{2}}(1-c_{\mathrm{R}}^2/\beta^2)^{\frac{1}{2}}(1-x_2^2/|x|^2)}{(2c_{\mathrm{R}}|x|)^{\frac{1}{2}}\beta^2 c_{\mathrm{R}}^2\mu D_\omega}\,\mathrm{Re}\left\{\left(t-\frac{|x|}{c_{\mathrm{R}}}+0\mathrm{i}\right)^{-\frac{3}{2}}\right\} \tag{2.6.25}$$

when $x_3 = 0$ and $t \to |x|/c_{\mathrm{R}}$. Not unexpectedly, this vanishes when $x_1 = 0$, corresponding to the fact that the surface displacement of a Rayleigh wave has no 'transverse' component. The result (2.6.25) yields a 'radiation pattern' which is consistent with one derived by Cherry (1962), from a solution of the corresponding 'time-harmonic' problem.

The two-dimensional traction problem may be specialized similarly. It is obtained, for example, from (2.5.11), that

$$U_{33}(t,x) = \frac{1}{2\pi^2\mu}\,\mathrm{Im}\left\{\frac{-\left(\dfrac{\Omega_\alpha^2}{\alpha^2}-1\right)^{\frac{1}{2}}\left(2-\dfrac{\Omega_\alpha^2}{\beta^2}\right)H\left(t-\dfrac{x_3}{\alpha}\right)}{D(\Omega_\alpha,1)\left(t-\dfrac{\Omega_\alpha}{\alpha^2}\left(\dfrac{\Omega_\alpha^2}{\alpha^2}-1\right)^{-\frac{1}{2}}x_3\right)}+\frac{2\left(\dfrac{\Omega_\beta^2}{\alpha^2}-1\right)^{\frac{1}{2}}H\left(t-\dfrac{x_3}{\beta}\right)}{D(\Omega_\beta,1)\left(t-\dfrac{\Omega_\beta}{\beta^2}\left(\dfrac{\Omega_\beta^2}{\beta^2}-1\right)^{-\frac{1}{2}}x_3\right)}\right\}, \tag{2.6.26}$$

where Ω_α, Ω_β are defined by equations (2.6.12) with $\eta = \nu$. The body-wave arrivals for $U_{33}(t, x)$ may now be obtained either directly from (2.2.26), or from (2.5.12); either method gives

$$U_{33}(t,x) \sim \frac{1}{2\pi^2\mu}\,\mathrm{Im}\left\{\left[2(\nu_\lambda x_\lambda)^2 - \frac{\alpha^2}{\beta^2}|x|^2\right]x_3^2\Big/\left[\left(\frac{2|x|}{\alpha}\right)^{\frac{1}{2}}\left(\frac{|x|}{\alpha}-t\right)^{\frac{1}{2}} D(-\alpha|x|, \nu_\lambda x_\lambda)\right]\right\}, \quad (2.6.27)$$

as $t \to |x|/\alpha$, and

$$U_{33}(t,x) \sim \frac{1}{\pi^2\mu}\,\mathrm{Im}\left\{\left(\frac{\beta^2|x|^2}{\alpha^2(\nu_\lambda x_\lambda)^2} - 1\right)^{\frac{1}{2}}\left[\frac{2|x|}{\beta}\left(\frac{|x|}{\beta}-t\right)\right]^{-\frac{1}{2}}\Big/ D\left(-\frac{\beta|x|}{\nu_\lambda x_\lambda}, 1\right)\right\}, \quad (2.6.28)$$

as $t \to |x|/\beta$. The expression (2.6.28) also displays a 'conical wave', due to the change in its analytic form across the cone $\beta^2|x|^2 = \alpha^2(\nu_\lambda x_\lambda)^2$. The Rayleigh arrival of $U(t, x)$ can be found from (2.6.13), (2.6.14) and (2.5.13), if $\nu_\lambda x_\lambda < 0$; for example,

$$U_{13}(t,x) \sim \frac{\nu_1\left[2 - \frac{c_R^2}{\beta^2} - 2\left(1-\frac{c_R^2}{\alpha^2}\right)^{\frac{1}{2}}\left(1-\frac{c_R^2}{\beta^2}\right)^{\frac{1}{2}}\right]}{2\pi^2\mu c_R^3 D_\omega(c_R t - |\nu_\lambda x_\lambda|)}, \quad (2.6.29)$$

$$U_{33}(t,x) \sim \frac{1}{2\pi^2\mu c_R^3 D_\omega}\frac{c_R^2}{\beta^2}\left(1-\frac{c_R^2}{\alpha^2}\right)^{\frac{1}{2}}\delta(c_R t - |\nu_\lambda x_\lambda|), \quad (2.6.30)$$

when $x_3 = 0$ and $t \to |\nu_\lambda x_\lambda|/c_R$, where D_ω is given by equation (2.6.24).

3. The homogeneous displacement boundary-value problem

3.1. *The interfacial dislocation*

The problem dual to that of §2.1 is defined by the equation of motion (2.1.3), the initial conditions (2.1.4) and the boundary condition that the surface displacement is a given homogeneous function of degree $n+1$ in (t, x_α). However, this problem itself is a special case of one for two dissimilar half-spaces, $x_3 > 0$ with elastic moduli c^+_{ijkl} and $x_3 < 0$ with elastic moduli c^-_{ijkl}, on whose interface there is a dislocation of strength $b(t, x_\alpha)$, where $b(t, x_\alpha)$ is a homogeneous function of degree $n+1$. This dislocation problem is defined by equations of motion like (2.1.3) in either half-space, the initial conditions (2.1.4) (for $-\infty < x_3 < \infty$) and the boundary conditions

$$\left.\begin{aligned} [u^+(t,x) - u^-(t,x)]_{x_3=0} &= b(t, x_\alpha),\\ u^\pm(t,x) \to 0, \quad |x_3| &\to \infty,\\ c^+_{i3kl}u^+_{k,l}|_{x_3=0} = c^-_{i3kl}u^-_{k,l}|_{x_3=0} &= T_i(t, x_\alpha) \quad \text{say,}\end{aligned}\right\} \quad (3.1.1)$$

where $u^+(t, x)$ is the displacement in the half-space $x_3 > 0$ and $u^-(t, x)$ that in the half-space $x_3 < 0$.

The interfacial traction vector $T(t, x_\alpha)$ is not at the moment known but, proceeding as in §2.1, the Fourier transforms $\tilde{u}^\pm(\omega, \xi_\alpha, x_3)$ may be expressed in terms of its Fourier transform $T(\omega, \xi_\alpha)$ as

$$\left.\begin{aligned} \tilde{u}^+(\omega, \xi_\alpha, x_3) &= (2\pi)^{\frac{3}{2}}\sum_{N=4}^{6} B^{+N}(\omega, \xi_\alpha)\exp\{-\mathrm{i}\xi_3^N(\omega,\xi_\alpha)x_3\}\,\tilde{T}(\omega,\xi_\alpha),\\ \tilde{u}^-(\omega, \xi_\alpha, x_3) &= (2\pi)^{\frac{3}{2}}\sum_{N=1}^{6} B^{-N}(\omega, \xi_\alpha)\exp\{-\mathrm{i}\xi_3^N(\omega,\xi_\alpha)x_3\}\,\tilde{T}(\omega,\xi_\alpha),\end{aligned}\right\} \quad (3.1.2)$$

where the matrices $B^{+N}(\omega, \xi_\alpha)$, $B^{-N}(\omega, \xi_\alpha)$ are as in equation (2.1.21), but with c_{ijkl} replaced by c^+_{ijkl}, c^-_{ijkl} rsepectively. The sum in the second of equations (3.1.2) ranges over $N = 1, 2, 3$ so that the transform remains finite as $x_3 \to -\infty$ when ω has positive imaginary part. Now by employing the Fourier transform of $(3.1.1)_1$, it is obtained that

$$\tilde{b}(\omega, \xi_\alpha) = (2\pi)^{\frac{3}{2}} A(\omega, \xi_\alpha)\, \tilde{T}(\omega, \xi_\alpha), \tag{3.1.3}$$

where
$$A(\omega, \xi_\alpha) = \sum_{N=4}^{6} B^{+N}(\omega, \xi_\alpha) - \sum_{N=1}^{3} B^{-N}(\omega, \xi_\alpha) \tag{3.1.4}$$

and $\tilde{b}(\omega, \xi_\alpha)$ denotes the Fourier transform of $b(t, x_\alpha)$. Therefore,

$$\tilde{T}(\omega, \xi_\alpha) = (2\pi)^{-\frac{3}{2}} A^{-1}(\omega, \xi_\alpha)\, \tilde{b}(\omega, \xi_\alpha), \tag{3.1.5}$$

and
$$\left.\begin{aligned} \tilde{u}^{+}(\omega, \xi_\alpha, x_3) &= \sum_{N=4}^{6} B^{+N}(\omega, \xi_\alpha)\, A^{-1}(\omega, \xi_\alpha)\, \tilde{b}(\omega, \xi_\alpha) \exp\{-\mathrm{i}\xi_3^N(\omega, \xi_\alpha)\, x_3\}, \\ \tilde{u}^{-}(\omega, \xi_\alpha, x_3) &= \sum_{N=1}^{3} B^{-N}(\omega, \xi_\alpha)\, A^{-1}(\omega, \xi_\alpha)\, \tilde{b}(\omega, \xi_\alpha) \exp\{-\mathrm{i}\xi_3^N(\omega, \xi_\alpha)\, x_3\}. \end{aligned}\right\} \tag{3.1.6}$$

Now exploiting the homogeneity of $b(t, x_\alpha)$, its Fourier transform can be calculated as

$$\tilde{b}(\omega, \xi_\alpha) = -(2\pi)^{-\frac{3}{2}}(-\mathrm{i})^n\, \partial_\omega^{(n+3)} \int \frac{\check{b}(p, \xi_\alpha)\, \mathrm{d}p}{\omega + p}, \tag{3.1.7}$$

where the Radon transform $\check{b}(p, \xi_\alpha)$ is defined by the equation

$$\check{b}(p, \xi_\alpha) = \iint b(1, y_\alpha)\, \delta(p - \xi_\alpha y_\alpha)\, \mathrm{d}y_1\, \mathrm{d}y_2. \tag{3.1.8}$$

The transform (3.1.6) can now be reduced by exactly the method that was applied to the expression (2.1.20) to yield

$$\partial_t^{(n+2)} u^{+}(t, x) = -\frac{\mathrm{i}}{2\pi} \sum_{N=4}^{6} \oint_{|\eta|=1} \mathrm{d}s \frac{(\Omega^N)^{n+2} B^{+N}(\Omega^N, \eta_\alpha)\, A^{-1}(\Omega^N \eta_\alpha)}{t + \xi_{3,\Omega}^N(\Omega^N, \eta)\, x_3} F^{(n+3)}(-\Omega^N, \eta)\, H(t - t_N), \tag{3.1.9}$$

where now
$$F^{(n+3)}(z, \eta) = \frac{1}{2\pi\mathrm{i}} \partial_z^{(n+3)} \int \frac{\check{b}(p, \eta_\alpha)\, \mathrm{d}p}{p - z}. \tag{3.1.10}$$

The corresponding expression for $\partial_t^{(n+2)} u^{-}(t, x)$ is similar. The right side of (3.1.9) can again be shown to be real, by the argument that was applied to equation (2.1.35). The $(n+1)$th derivative of the stress can be derived similarly. For example, if $\tau^{+}(t, x)$ denotes the column vector with components $\sigma_{i3}^{+}(t, x)$, then

$$\begin{aligned} \partial_t^{(n+1)} \tau^{+}(t, x) = -\frac{\mathrm{i}}{2\pi} \sum_{N=4}^{6} \oint_{|\eta|=1} \mathrm{d}s & \frac{(\Omega^N)^{n+1} C^{+N}(\Omega^N, \eta)\, B^{+N}(\Omega^N, \eta_\alpha)\, A^{-1}(\Omega^N, \eta_\alpha)}{t + \xi_{3,\Omega}^N(\Omega^N, \eta)\, x_3} \\ & \times F^{(n+3)}(-\Omega^N, \eta)\, H(t - t_N), \end{aligned} \tag{3.1.11}$$

where
$$C^{+N}(\Omega^N, \eta) = C^{+}(\xi^N); \quad \xi^N = (\eta_1, \eta_2, \xi_3^N(\Omega^N, \eta)). \tag{3.1.12}$$

When x_3 tends to zero, $\tau^{+}(t, x)$ reduces to the interfacial traction vector $T(t, x_\alpha)$ and (3.1.11) gives

$$\partial_t^{(n+1)} T(t, x_\alpha) = \frac{(2\pi)^{-\frac{5}{2}}}{t} \oint_{|\eta|=1} \mathrm{d}s\, \Omega^{n+1} A^{-1}(\Omega + 0\mathrm{i}, \eta_\alpha)\, F^{(n+3)}(-\Omega - 0\mathrm{i}, \eta), \tag{3.1.13}$$

in which $\Omega = -\eta_\alpha x_\alpha / t$, since from equation (2.1.21),

$$\sum_{N=4}^{6} C^{+N}(\Omega, \eta)\, B^{+N}(\Omega, \eta) = \mathrm{i}(2\pi)^{-\frac{3}{2}} I. \tag{3.1.14}$$

Equation (3.1.13) could also be obtained directly from (3.1.5).

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The body-wave arrivals associated with (3.1.9) are obtainable directly from the results of § 2.2; it is necessary only to set

$$\tilde{T}(\omega, \xi_\alpha) = -(2\pi)^{-3}(-\mathrm{i})^n A^{-1}(\omega, \xi_\alpha)\, \partial_\omega^{(n+3)} \int \frac{\check{b}(p, \xi_\alpha)\, \mathrm{d}p}{p+\omega} \tag{3.1.15}$$

in the expressions (2.2.15), (2.2.19) and (2.2.26). There is, of course, no Rayleigh arrival because the matrix $A^{-1}(\omega, \xi_\alpha)$ will be zero whenever

$$B^+ = \sum_{N=4}^{6} B^{+N} \quad \text{or} \quad B^- = \sum_{N=1}^{3} B^{-N}$$

is singular.

3.2. *The two-dimensional interfacial dislocation*

In the formulation of § 3.1, it was assumed that the integral (3.1.8) exists. This is not true for the special case in which

$$b(t, x_\alpha) = t^{n+1} b(p), \tag{3.2.1}$$

where

$$p = \nu_\lambda x_\lambda / t, \tag{2.3.1}$$

but representations for $u^+(t, x)$ and $u^-(t, x)$ for this case can be obtained from the results of § 2.3, once an expression for $\tilde{T}(\Omega, \eta_\alpha)$ has been developed. First, a calculation of $\tilde{b}(\Omega, \eta_\alpha)$ like the one which led to equation (2.3.5) gives

$$\tilde{b}(\Omega, \eta_\alpha) = (-\mathrm{i})^{n+1}(2\pi)^{-\frac{1}{2}}\, \partial_\Omega^{(n+2)} \left\{ \delta_1(\lambda_\alpha \eta_\alpha) \int \frac{b(p)\, \mathrm{d}p}{p+\Omega} - \delta_2(\lambda_\alpha \eta_\alpha) \int \frac{b(p)\, \mathrm{d}p}{p-\Omega} \right\}, \tag{3.2.2}$$

so that, from (3.1.5), $\tilde{T}(\Omega, \eta_\alpha)$ is (3.2.2) premultiplied by $(2\pi)^{-\frac{3}{2}} A^{-1}(\Omega, \eta_\alpha)$. This is similar in form to (2.3.5) and so the reduction corresponding to (2.3.7) can be carried out at sight to give

$$\partial_t^{(n+2)} u^+(t, x) = 2(2\pi)^{-3}(-1)^n \sum_{N=4}^{6} \mathrm{Im} \left\{ \frac{(\Omega^N)^{n+2} B^{+N}(\Omega^N, \nu_\alpha)\, A^{-1}(\Omega^N, \nu_\alpha)}{t + \xi_{3,\Omega}^N(\Omega^N, \nu)\, x_3}\, \partial_{\Omega^N}^{(n+2)} \int \frac{b(p)\, \mathrm{d}p}{p+\Omega^N} \right\} H(t - t_N), \tag{3.2.3}$$

where $\Omega^N = \Omega^N(\nu)$. Thus; if $G^{(n+2)}(z, \eta)$ is defined as

$$G^{(n+2)}(z, \eta) = \frac{1}{2\pi \mathrm{i}}\, \partial_z^{(n+2)} \int \frac{b(p)\, \mathrm{d}p}{p-z}, \tag{3.2.4}$$

equation (3.2.3) becomes

$$\partial_t^{(n+2)} u^+(t, x) = -2(2\pi)^{-2} \sum_{N=4}^{6} \mathrm{Re} \left\{ \frac{(\Omega^N)^{n+2} B^{+N}(\Omega^N, \nu_\alpha)\, A^{-1}(\Omega^N, \nu_\alpha)}{t + \xi_{3,\Omega}^N(\Omega^N, \nu)\, x_3}\, G^{(n+2)}(-\Omega^N, \nu) \right\} H(t - t_N), \tag{3.2.5}$$

which is similar in form to (2.3.9).

The $(n+1)$th derivative of the traction vector $\tau^+(t, x)$ which was introduced in the preceding section can be calculated similarly as

$$\partial_t^{(n+1)} \tau^+(t, x) = -2(2\pi)^{-2} \sum_{N=4}^{6} \mathrm{Re} \left\{ \frac{(\Omega^N)^{n+1} C^{+N}(\Omega^N, \nu)\, B^{+N}(\Omega^N, \nu_\alpha)\, A^{-1}(\Omega^N, \nu_\alpha)}{t + \xi_{3,\Omega}^N(\Omega^N, \nu)\, x_3}\, G^{(n+2)}(-\Omega^N, \nu) \right\} H(t - t_N). \tag{3.2.6}$$

When x_3 tends to zero, (3.2.6) reduces to

$$\partial_t^{(n+1)} T(t, x_\alpha) = \frac{2(2\pi)^{-\frac{7}{2}}(-1)^n}{t}\, \mathrm{Im}\, \{p^{n+1} A^{-1}(-p + 0\mathrm{i}, \nu_\alpha)\, G^{(n+2)}(p - 0\mathrm{i}, \nu)\}, \tag{3.2.7}$$

where p is given by (2.3.1), since $\Omega^N \rightarrow -p + 0\mathrm{i}$ and (3.1.14) holds.

The body-wave arrivals associated with (3.2.5) can be found by the method of § 2.4; the result which corresponds to (2.4.16) is

$$\partial_t^{(n+2)} u^+(t,x) \sim -2(2\pi)^{-2}\,\mathrm{Re}\,\{(\Omega^0)^{n+2}\,B^{+N}(\Omega^0,\nu_\alpha)\,A^{-1}(\Omega^0,\nu_\alpha)\,G^{(n+2)}(-\Omega^0,\nu) \times [2\xi^N_{3,\Omega\Omega}(1,s^0\nu)\,x_3(t_0-t)]^{-\frac{1}{2}}\}\,\mathrm{sgn}\,[\xi^N_{3,\Omega\Omega}(1,s^0\nu)], \quad (3.2.8)$$

as $t \to t_0$, the notation being that employed in § 2.4.

4. Some simple mixed boundary-value problems

4.1. *A generalization of Boussinesq's problem*

For the half-space $x_3 > 0$, suppose that the displacement field satisfies the equation of motion (2.1.3), the initial conditions (2.1.4) and the mixed boundary conditions

$$\left.\begin{aligned} c_{\alpha 3kl}u_{k,l}(t,x) &= 0, \quad (x_3 = 0, \quad \text{all} \quad x_1, x_2, \quad \alpha = 1,2),\\ c_{33kl}u_{k,l}(t,x) &= 0 \quad (x_3 = 0, \quad x \notin S(t)),\\ u_3(t,x) &= w_{n+1}(t,x_\alpha) \quad (x \in S(t)),\\ u(t,x) &\to 0; \quad (|x| \to \infty), \end{aligned}\right\} \quad (4.1.1)$$

where $S(t)$ is the surface $$x_3 = 0, \quad s_1^2x_1^2 + s_2^2x_2^2 \leqslant t^2 \quad (4.1.2)$$

and is 'subsonic' in the sense of § 2.2, and $w_{n+1}(t,x_\alpha)$ is a homogeneous polynomial of degree $n+1$ in (t,x_α). This problem is a dynamical extension of the problem of the static indentation of a half-space by a smooth punch, the prototype of which was solved by Boussinesq in 1885 (see, for instance, Love 1944). The original static problem appears as a limiting case of the one defined above, when the density ρ of the half-space is made to tend to zero, so that the equation of motion (2.1.3) reduces to the equation of equilibrium and the role of the variable t reduces to that of a parameter. The boundary conditions (4.1.1) imply that the surface traction $T(t,x_\alpha)$ has only one non-zero component, $T_3(t,x_\alpha)$ and that its support is $S(t)$. If $T_3(t,x_\alpha)$ could be found, the problem would be solved immediately from the results of § 2, and the object of the present section is to set up and solve an integral equation for $T_3(t,x_\alpha)$. It will emerge that the structure of the integral equation is the same in the general case as in the static limit, so that a unified approach to either the static or dynamic problem is obtained. The solution of the full problem will then follow as a straightforward extension of the static solution of Willis (1967).

The simple structure of the problem is a consequence of Betti's theorem, which implies that

$$U(t,x) = U^{\mathrm{T}}(t,-x) \quad (x_3 = 0), \quad (4.1.3)$$

where $U(t,x)$ is the solution of Lamb's problem (that is, the Green matrix for surface loading of the half-space) and U^{T} denotes the transpose of U. Therefore, from the definition of $B(\omega,\xi_\alpha)$ as the Fourier transform of $U(t,x)$ with $x_3 = 0$,

$$B(\Omega,\eta_\alpha) = B^{\mathrm{T}}(\Omega,-\eta_\alpha). \quad (4.1.4)$$

Also, if Ω is real and $$|\Omega| < c(\eta_1,\eta_2,0) \quad (4.1.5)$$

for all wave speeds c, the equation $$|K(\Omega,\eta,\xi_3)| = 0 \quad (4.1.6)$$

has complex roots ξ_3^N and since only Ω^2 appears in (4.1.5),

$$\xi_3^N(\Omega+0\mathrm{i},\eta) = \xi_3^M(-\Omega+0\mathrm{i},\eta) \quad (4.1.7)$$

for some M not necessarily equal to N but still in the range (4, 5, 6) if N is in this range, since then ξ_3^M has negative imaginary part. If the restriction (4.1.4) does not hold, (4.1.7) remains valid but then, since ξ_3^N could be real, M need not lie in the range (4, 5, 6) whenever N does. Now from (4.1.7) and the definition (2.1.21) of B^N,

$$B^N(\Omega+0\mathrm{i},\eta_\alpha) = B^M(-\Omega+0\mathrm{i},\eta_\alpha) \tag{4.1.8}$$

and so, whenever (4.1.5) holds, (4.1.4) and (4.1.8) imply

$$B(-\Omega+0\mathrm{i},-\eta_\alpha) = B^{\mathrm{T}}(\Omega+0\mathrm{i},\eta_\alpha). \tag{4.1.9}$$

Also, the argument following equation (2.1.36) shows that

$$B(-\bar{\Omega},-\eta_\alpha) = \overline{B(\Omega,\eta_\alpha)} \tag{4.1.10}$$

for any Ω, and so

$$B(\Omega+0\mathrm{i},\eta_\alpha) = \overline{B^{\mathrm{T}}(\Omega+0\mathrm{i},\eta_\alpha)}. \tag{4.1.11}$$

Thus, the matrix $B(\Omega+0\mathrm{i},\eta_\alpha)$ is hermitian. In particular, $B_{33}(\Omega+0\mathrm{i},\eta_\alpha)$ is a real and even function, both of Ω and η_α. All of these properties can, of course, be recognized explicitly from equations (2.6.13) and (2.6.14) which relate to an isotropic half-space.

To proceed now with the solution, the boundary conditions (4.1.1) define a field $u(t,x)$ which is homogeneous of degree $n+1$. Therefore, $T(t,x_\alpha)$ is homogeneous of degree n and the representation (2.1.35), implies

$$\partial_t^{(n+2)}u_3(t,x) = -\frac{(2\pi)^{\frac{1}{2}}}{t}\oint_{|\eta|=1}\mathrm{d}s\,\mathrm{Re}\left\{\left(-\frac{\eta_\lambda x_\lambda}{t}\right)^{n+2}B_{33}\left(-\frac{\eta_\lambda x_\lambda}{t}+0\mathrm{i},\eta_\alpha\right)F_3^{(n+2)}\left(\frac{\eta_\lambda x_\lambda}{t}-0\mathrm{i},\eta\right)\right\}, \tag{4.1.12}$$

when $x_3=0$, where

$$F_3^{(n+2)}(z,\eta) = \frac{1}{2\pi\mathrm{i}}\,\partial_z^{(n+2)}\int\frac{\check{T}_3(p,\eta)\,\mathrm{d}p}{p-z}, \tag{4.1.13}$$

since $T_1=T_2=0$. The insertion of the 'real part' sign was justified after equation (2.1.36). Now when $x\in S(t)$, $B_{33}(-\eta_\lambda x_\lambda/t+0\mathrm{i},\eta_\alpha)$ is real, by (4.1.11). Since $\check{T}_3(p,\eta)$ is real, it now follows from the Plemelj formulae (Muskhelishvili 1953) that

$$\partial_t^{(n+2)}u_3(t,x) = \frac{1}{t}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\oint_{|\eta|=1}\mathrm{d}s\left(-\frac{\eta_\lambda x_\lambda}{t}\right)^{n+2}B_{33}\left(-\frac{\eta_\lambda x_\lambda}{t}+0\mathrm{i},\eta_\alpha\right)\check{T}_3^{(n+2)}\left(\frac{\eta_\lambda x_\lambda}{t},\eta\right), \tag{4.1.14}$$

when $x\in S(t)$, where

$$\check{T}_3^{(n+2)}(\Omega,\eta) = \partial_\Omega^{(n+2)}\check{T}_3(\Omega,\eta). \tag{4.1.15}$$

The remarkably simple representation (4.1.14) that is provided by use of the Radon transform has not been given before for a dynamical problem, but a similar representation for a related static problem was given by Willis (1970). The only change that appears in (4.1.14) in the static limit ($\rho\to 0$) is that B_{33} reduces to a function of η_α only, since the first argument is always combined with the density ρ.

It can now be observed that, if $\check{T}_3(\Omega,\eta)$ is a polynomial of degree $n+1$ in Ω, then

$$\partial_t^{(n+2)}u(t,x) = 0, \tag{4.1.16}$$

when $x\in S(t)$. A representation analogous to (4.1.14) can also be found for any derivative of order $n+2$ of $u(t,x)$. The steps are identical with those given for the case already worked out and give the result

$$\partial_t^p\,\partial_{x_1}^q\,\partial_{x_2}^r u_3(t,x) = \frac{1}{t}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\oint_{|\eta|=1}\mathrm{d}s\left(-\frac{\eta_\lambda x_\lambda}{t}\right)^p\eta_1^q\eta_2^r B_{33}\left(-\frac{\eta_\lambda x_\lambda}{t}+0\mathrm{i},\eta_\alpha\right)\check{T}_3^{(n+2)}\left(\frac{\eta_\lambda x_\lambda}{t},\eta\right), \tag{4.1.17}$$

when $x \in S(t)$ and $p+q+r = n+2$. This also vanishes when $\check{T}_3(\Omega, \eta)$ is a polynomial of degree $n+1$ in Ω. In this case, therefore, all of the derivatives of order $n+2$ of $u_3(t, x_1, x_2, 0)$ are zero and hence $u_3(t, x)$ is a polynomial of degree $n+1$ in (t, x_α) when $x \in S(t)$. Now an explicit calculation was performed by Willis (1970) to show that the function

$$T_3(y) = (s_1 y_1 + \mathrm{i} s_2 y_2)^P (s_1 y_1 - \mathrm{i} s_2 y_2)^Q (1 - s_1^2 y_1^2 - s_2^2 y_2^2)^{-\frac{1}{2}} \tag{4.1.18}$$

has Radon transform

$$\check{T}_3(\Omega, \eta) = \frac{\pi}{s_1 s_2}\left(\frac{\eta_1^2}{s_1^2}+\frac{\eta_2^2}{s_2^2}\right)^{-\frac{1}{2}} H(1-v^2)\,(\lambda_1 + \mathrm{i}\lambda_2)^{P-Q} \times \sum_{M=0}^{P}\sum_{N=0}^{Q}(-1)^{\frac{1}{2}(M+N)}\binom{P}{M}\binom{Q}{N}\binom{M+N}{\frac{1}{2}(M+N)}\left(\frac{1-v^2}{4}\right)^{\frac{1}{2}(M+N)} v^{P+Q-M-N}, \tag{4.1.19}$$

where
$$\lambda_1 + \mathrm{i}\lambda_2 = \left(\frac{\eta_1}{s_1}+\frac{\mathrm{i}\eta_2}{s_2}\right)\left(\frac{\eta_1^2}{s_1^2}+\frac{\eta_2^2}{s_2^2}\right)^{-\frac{1}{2}} \tag{4.1.20}$$

and
$$v = \Omega\left(\frac{\eta_1^2}{s_1^2}+\frac{\eta_2^2}{s_2^2}\right)^{-\frac{1}{2}}. \tag{4.1.21}$$

Therefore, since any $T_3(y)$ of the form

$$T_3(y) = P_{n+1}(y)\,(1 - s_1^2 y_1^2 - s_2^2 y_2^2)^{-\frac{1}{2}}, \tag{4.1.22}$$

where P_{n+1} is a polynomial of degree $n+1$, can be expressed as a sum of terms like (4.1.18), the traction

$$T_3(t, x_\alpha) = t^n P_{n+1}(y)\,(1 - s_1^2 y_1^2 - s_2^2 y_2^2)^{-\frac{1}{2}}, \tag{4.1.23}$$

which is homogeneous of degree n in (t, x_α), produces a normal displacement $u_3(t, x)$ over $S(t)$ which is a homogeneous polynomial of degree $n+1$ in (t, x_α). Further, the form (4.1.23) contains sufficient arbitrary constants to generate any homogeneous polynomial $u_3(t, x_\alpha)$, so that the solution to the problem (4.1.1) always has the form (4.1.23). Finally, since any polynomial can be expressed as a sum of homogeneous polynomials, this restriction may now be dropped and the following result is obtained:

The traction

$$T_3(t, x_\alpha) = P_{n+1}(t, x_\alpha)\,(t^2 - s_1^2 x_1^2 - s_2^2 x_2^2)^{-\frac{1}{2}} H(t^2 - s_1^2 x_1^2 - s_2^2 x_2^2) \tag{4.1.24}$$

produces a normal displacement of the form

$$u_3(t, x) = Q_{n+1}(t, x_\alpha) \tag{4.1.25}$$

over $S(t)$, where $P_{n+1}(t, x_\alpha)$, $Q_{n+1}(t, x_\alpha)$ are polynomials of degree $n+1$. This result, which applies for a half-space of any anisotropy, generalizes the corresponding static result of Willis (1967), which was itself a generalization of a result for an isotropic half-space, termed 'Galin's theorem' by Sneddon (1966). Of course, if $Q_{n+1}(t, x_\alpha)$ is specified, there remains the task of finding the coefficients in $P_{n+1}(t, x_\alpha)$. The problem has thus been reduced to one of solving a set of linear algebraic equations, which are obtained by integrating (4.1.12) for a traction T_3 of the form (4.1.23). This will not be discussed further here, but an example will be worked out in § 4.4 below.

4.2. *The two-dimensional problem*

A similar development may be given for the corresponding two-dimensional problem, in which $S(t)$ is the strip

$$|\nu_\lambda x_\lambda| \leqslant Vt, \tag{4.2.1}$$

where V is subsonic, and $w_{n+1}(t, x_\alpha)$ depends only on $(t, \nu_\alpha x_\alpha)$, so that

$$w_{n+1}(t, x_\alpha) = t^{n+1} w(p), \tag{4.2.2}$$

where p is defined by equation (2.3.1). Although $w_{n+1}(t, x_\alpha)$ is required to be homogeneous of degree $n+1$, the restriction to polynomial form will be relaxed, so that $w(p)$ may be any sufficiently smooth function. In this case, $T_1 = T_2 = 0$ and

$$T_3(t, x_\alpha) = t^n T_3(p), \tag{4.2.3}$$

where the unknown function $T_3(p)$ is zero for $|p| > V$. An integral equation for $T_3(p)$ can be developed from equation (2.3.9) which gives, when $x_3 = 0$,

$$\partial_t^{(n+2)} u_3(t, x) = \frac{2(2\pi)^{-\frac{1}{2}}}{t} \operatorname{Im}\left\{\left(-\frac{\nu_\lambda x_\lambda}{t}\right)^{n+2} B_{33}\left(-\frac{\nu_\lambda x_\lambda}{t} + 0\mathrm{i}, \nu_\alpha\right) G_3^{(n+1)}\left(\frac{\nu_\lambda x_\lambda}{t} - 0\mathrm{i}, \nu\right)\right\}, \tag{4.2.4}$$

where
$$G_3^{(n+1)}(z, \nu) = \frac{1}{2\pi\mathrm{i}} \partial_z^{(n+2)} \int \frac{T_3(p)\,\mathrm{d}p}{p-z}. \tag{4.2.5}$$

When $x \in S(t)$, $u_3(t, x) = w_{n+1}(t, x_\alpha)$ and $\partial_t^{(n+2)} u_3(t, x_\alpha)$ has the form

$$\partial_t^{(n+2)} u_3(t, x) = w^{(n+2)}(p)/t, \tag{4.2.6}$$

where
$$w^{(n+2)}(p) = \partial_t^{(n+2)} u_3(1, y) \quad (y \in S(1)).$$

Also, when $x \in S(t)$, B_{33} is real and so (4.1.29) furnishes the Hilbert problem

$$w^{(n+2)}(p) = -\mathrm{i}(2\pi)^{-\frac{1}{2}}(-p)^{n+2} B_{33}(-p+0\mathrm{i}, \nu_\alpha)\{G_3^{(n+1)}(p+0\mathrm{i}, \nu) + G_3^{(n+1)}(p-0\mathrm{i}, \nu)\} \quad (|p| < V), \tag{4.2.7}$$

for $G_3^{(n+1)}(z, \nu)$. From its definition (4.1.30), $G_3^{(n+1)}(z, \nu)$ is holomorphic in the z-plane cut along the interval $(-V, V)$ of the real axis, is $O(z^{-n-2})$ as $|z| \to \infty$ and has singularities no worse than $(z \mp V)^{-n-2+\delta}$ at $z = \pm V$, where $\delta > 0$, provided that

$$T_3(p) \sim A(V-p)^{\delta-1},$$

as $p \to V$, with a similar condition as $p \to -V$. The general solution of the Hilbert problem (4.2.7) may now be deduced (Muskhelishvili 1953) as

$$G_3^{(n+1)}(z, \nu) = (-z)^{-n-2}\left\{(-\mathrm{i})\,(2\pi)^{-\frac{1}{2}}\,(z^2 - V^2)^{\frac{1}{2}} \int_{-V}^{V} \frac{w^{(n+2)}(p)\,\mathrm{d}p}{B_{33}(-p+0\mathrm{i}, \nu_\alpha)\,(V^2-p^2)^{\frac{1}{2}}\,(p-t)} + P_{2n+3}(z)\,(z^2 - V^2)^{-n-\frac{3}{2}}\right\}, \tag{4.2.8}$$

where $P_{2n+3}(z)$ is a polynomial of degree $2n+3$ in z, whose coefficients must be chosen so that $G_3^{(n+1)}(z, \nu)$ is finite at $z = 0$ but otherwise are arbitrary and thus are available for adjusting $u_3(t, x)$ which is obtained by integration of (4.2.4), to its required form. Although knowledge of $G_3^{(n+1)}(z, \nu)$ is sufficient for the calculation of $\partial_t^{(n+2)} u(t, x)$ throughout the half-space, $T_3(p)$ itself will normally be required. To find $T_3(p)$, it is necessary first to integrate $G_3^{(n+1)}(z, \nu)$ $n+1$ times with respect to z to give $G_3(z, \nu)$, from which $T_3(p)$ follows from the Plemelj formulae as

$$T_3(p) = G_3(p+0\mathrm{i}, \nu) - G_3(p-0\mathrm{i}, \nu). \tag{4.2.9}$$

A detailed example will be given in § 4.4. This section will now be concluded with a brief general discussion of the case where $w_{n+1}(t, x_\alpha)$ is a polynomial, so that $w^{(n+2)}(p) = 0$. $G_3^{(n+1)}(z, \nu)$ is now finite at the origin if $P_{2n+3}(z)$ contains z^{n+2} as a factor. Thus,

$$G_3^{(n+1)}(z, \nu) = P_{n+1}(z)\,(z^2 - V^2)^{-n-\frac{3}{2}}, \tag{4.2.10}$$

where $P_{n+1}(z)$ is a polynomial of degree $n+1$. Integrating (4.2.10) now leads to a result of the form

$$G_3(z, \nu) = Q_{n+1}(z)\,(z^2 - V^2)^{-\frac{1}{2}} + R_n(z), \tag{4.2.11}$$

where Q_{n+1}, R_n are polynomials of the degree indicated, so that, from (4.2.9),

$$T_3(p) = -2\mathrm{i}Q_{n+1}(p)\,(V^2 - p^2)^{-\frac{1}{2}} \quad (|p| < V). \tag{4.2.12}$$

Thus, the traction $T_3(t, x_\alpha) = t^n T_3(p)$ produces a normal displacement over $S(t)$ which is a homogeneous polynomial of degree $n+1$ in $(t, \nu_\lambda x_\lambda)$. Finally, the restriction to homogeneous polynomials may be relaxed to give a result of the same form as that obtained for the three-dimensional problem:

The traction $$T_3(t, x_\alpha) = P_{n+1}(t, \nu_\alpha x_\alpha)\,(V^2t^2 - (\nu_\alpha x_\alpha)^2)^{-\frac{1}{2}} H(Vt - |\nu_\alpha x_\alpha|) \tag{4.2.13}$$

produces a normal displacement over $S(t)$ of the form

$$u_3(t, x) = Q_{n+1}(t, \nu_\alpha x_\alpha), \tag{4.2.14}$$

where P_{n+1}, Q_{n+1} are polynomials of degree $n+1$ in their arguments.

4.3. *The axisymmetric 'smooth punch' problem*

If the half-space is isotropic or, more generally, transversely isotropic with the x_3-axis as its axis of symmetry, the function $B_{33}(\Omega + 0\mathrm{i}, \eta_\alpha)$ is independent of η_α (see equations (2.6.13) and (2.6.14) for an isotropic half-space), and an axisymmetric specialization of the problem discussed in § 4.1 can be defined. We take $S(t)$ to be the surface

$$x_3 = 0, \quad r \leqslant Vt, \tag{4.3.1}$$

where $r^2 = x_1^2 + x_2^2$ and V is subsonic, and let

$$u_3(t, x) = t^{n+1} w(r/t) \quad (x \in S(t)), \tag{4.3.2}$$

in which w is any sufficiently smooth function. With these restrictions, equations (2.1.3), (2.1.4) and (4.1.1) define an axisymmetric state of stress, and $T_3(t, x_\alpha)$ has the form

$$T_3(t, x_\alpha) = t^n T_*(q), \tag{4.3.3}$$

where $$q = r/t, \tag{4.3.4}$$

for some function T_*. The Radon transform $\check{T}_3(\Omega, \eta)$ is independent of η and can be expressed as

$$\check{T}_3(\Omega) = 2\int_\Omega^V \frac{T_*(q)\,q\,\mathrm{d}q}{(q^2 - \Omega^2)^{\frac{1}{2}}}. \tag{4.3.5}$$

Equation (4.3.5) is of Abel type and can be inverted immediately to give

$$T_*(q) = \frac{\check{T}_3(V)}{\pi(V^2 - q^2)^{\frac{1}{2}}} - \frac{1}{\pi}\int_q^V \frac{\mathrm{d}\check{T}_3(\Omega)}{\mathrm{d}\Omega}\,\frac{\mathrm{d}\Omega}{(\Omega^2 - q^2)^{\frac{1}{2}}}, \tag{4.3.6}$$

which can also be obtained, with a little more effort, from the general inversion formula (A 5).

Equation (4.1.14) also reduces to an equation of Abel type, since η now appears in the integrand of the right side only in the combination $(\eta_\lambda x_\lambda/t)$. Thus, upon defining $w^{(n+2)}(q)$ by the relation

$$\partial_t^{(n+2)} u_3(t,x) = (1/t)\, w^{(n+2)}(q), \quad (x \in S(t)), \tag{4.3.7}$$

and setting

$$\Omega = q\cos\theta, \tag{4.3.8}$$

equation (4.1.14) takes the form

$$w^{(n+2)}(q) = 2(2\pi)^{\frac{1}{2}} \int_0^q \frac{\mathrm{d}\Omega}{(q^2-\Omega^2)^{\frac{1}{2}}} (-\Omega)^{n+2} B_{33}(-\Omega+0\mathrm{i})\, \check{T}_3^{(n+2)}(\Omega). \tag{4.3.9}$$

This can be inverted to give

$$(2\pi)^{\frac{1}{2}}(-\Omega)^{n+2} B_{33}(-\Omega+0\mathrm{i})\, \check{T}_3^{(n+2)}(\Omega) = \frac{w^{(n+2)}(0)}{\pi} + \frac{\Omega}{\pi}\int_0^\Omega \frac{\mathrm{d}w^{(n+2)}(q)}{\mathrm{d}q} \frac{\mathrm{d}q}{(\Omega^2-q^2)^{\frac{1}{2}}}. \tag{4.3.10}$$

It is easy to show from its definition (4.3.7) that

$$w^{(n+2)}(q) = O(q^{n+2}) \tag{4.3.11}$$

as $q \to 0$. Hence

$$\check{T}_3^{(n+2)}(\Omega) = \frac{(-1)^{n+1}}{(2\pi)^{\frac{1}{2}}\pi B_{33}(-\Omega+0\mathrm{i})\,\Omega^{n+1}} \int_0^\Omega \frac{\mathrm{d}w^{(n+2)}(q)}{\mathrm{d}q} \frac{\mathrm{d}q}{(\Omega^2-q^2)^{\frac{1}{2}}}, \tag{4.3.12}$$

and is bounded since the integral in (4.3.12) is $O(\Omega^{n+1})$ as $\Omega \to 0$. The Radon transform $\check{T}_3(\Omega)$ follows by integrating (4.3.12) $n+2$ times, and is determined in this way up to a polynomial of degree $n+1$ in Ω, whose coefficients must be adjusted so that the displacement $u_3(t,x)$ takes the desired values over $S(t)$. This is effected by substituting $\check{T}_3(\Omega)$ into (4.1.12), integrating $n+2$ times with respect to t and then choosing the coefficients accordingly; the procedure is always possible because $w^{(n+2)}(q)$ determines $u_3(t,x)$ uniquely up to a homogeneous polynomial of degree $n+1$ in (t,r), when $x \in S(t)$.

It is interesting to note that the case $w^{(n+2)}(q) = 0$ is more than a particular case of the problem of § 4.1 since it includes, for example,

$$u_3(t,x) = \alpha r - Ut \quad (r \leqslant Vt), \tag{4.3.13}$$

which is not a polynomial in (t, x_α). For (4.3.13), $n = 0$ and (4.3.12) implies

$$\check{T}_3(\Omega) = C + D\Omega \tag{4.3.14}$$

for some C and D. Hence, from (4.3.6),

$$T_*(q) = \frac{C+DV}{\pi(V^2-q^2)^{\frac{1}{2}}} - \frac{D}{\pi}\ln\left\{\frac{V+(V^2-q^2)^{\frac{1}{2}}}{q}\right\}, \tag{4.3.15}$$

so that T_3 is singular when $r = Vt$ unless

$$C + DV = 0. \tag{4.3.16}$$

In this case,

$$T_3(t, x_\alpha) = \frac{-D}{\pi}\ln\left\{\frac{Vt+(V^2t^2-r^2)^{\frac{1}{2}}}{r}\right\} \tag{4.3.17}$$

and D and V may be adjusted to match α and U in (4.3.13). This solves the problem of indentation by a conical indentor, of semi-angle $\frac{1}{2}\pi - \alpha$, moving with speed U into the half-space. A detailed solution of this problem was given by Kostrov (1964*a*), but the method that was employed involved some ingenious guess-work and could not obviously be extended to deal with the general displacement (4.3.2).

4.4. *Examples*

(*a*) *The three-dimensional 'flat punch'* ($n = -1$)

As a simple illustration of the theory of § 4.1, consider the dynamical extension of Boussinesq's original problem, taking

$$u_3(t, x) = u_0, \quad x \in S(t), \tag{4.4.1}$$

where u_0 is a constant and $S(t)$ is the expanding ellipse (4.1.2). In this case, $n = -1$ and

$$T_3(t, x_\alpha) = T_0(t^2 - s_1^2 x_1^2 - s_2^2 x_2^2)^{-\frac{1}{2}} H(t^2 - s_1^2 x_1^2 - s_2^2 x_2^2), \tag{4.4.2}$$

where T_0 is a constant. The Radon transform $\check{T}_3(\Omega, \eta)$ is given from (4.1.19) as

$$\check{T}_3(\Omega, \eta) = \frac{T_0 \pi}{s_1 s_2 \omega_0(\eta)} H[(\omega_0(\eta))^2 - \Omega^2], \tag{4.4.3}$$

where
$$\omega_0(\eta) = \left(\frac{\eta_1^2}{s_1^2} + \frac{\eta_2^2}{s_2^2}\right)^{\frac{1}{2}}. \tag{4.4.4}$$

Correspondingly,
$$F_3'(z, \eta) = \frac{T_0}{\mathrm{i} s_1 s_2 [(\omega_0(\eta))^2 - z^2]}. \tag{4.4.5}$$

Hence, by taking the real part of (2.1.37) with $n = -1$,

$$u_i(t, x) = \frac{T_0 (2\pi)^{\frac{1}{2}}}{s_1 s_2} \sum_{N=4}^{6} \oint_{|\eta|=1} \mathrm{d}s \, \mathrm{Im} \left\{ \int_{\Omega^N(t, x, \eta)}^{\infty} \mathrm{d}\Omega \frac{B_{i3}^N(\Omega, \eta_\alpha)}{\omega_0^2 - \Omega^2} \right\}. \tag{4.4.6}$$

When $x_3 = 0$ and $i = 3$, (4.4.6) gives

$$u_3(t, x) = \frac{T_0 (2\pi)^{\frac{1}{2}}}{s_1 s_2} \oint_{|\eta|=1} \mathrm{d}s \, \mathrm{Im} \left\{ \int_{-\eta_\alpha x_\alpha / t}^{\infty} \mathrm{d}\Omega \frac{B_{33}(-\Omega + 0\mathrm{i}, \eta_\alpha)}{\omega_0^2 - (\Omega - 0\mathrm{i})^2} \right\}. \tag{4.4.7}$$

Now $B_{33}(-\Omega + 0\mathrm{i}, \eta_\alpha)$ is real when Ω is real and $|\Omega|$ is smaller than $c_\mathrm{R}(\eta)$, the speed of Rayleigh waves in the direction η. Hence, when $x \in S(t)$, the range of integration with respect to Ω can be split into two parts, $(-\eta_\alpha x_\alpha / t, c)$ and (c, ∞), in which $-\eta_\alpha x_\alpha / t < c < c_\mathrm{R}(\eta)$, and the integral over the former part can be reduced by the Plemelj formulae to give

$$u_3(t, x) = u_0 = \frac{T_0 (2\pi)^{\frac{1}{2}}}{s_1 s_2} \oint_{|\eta|=1} \mathrm{d}s \left[-\frac{\pi B_{33}(-\omega_0(\eta) + 0\mathrm{i}, \eta_\alpha)}{2\omega_0(\eta)} + \mathrm{Im} \left\{ \int_c^{\infty} \mathrm{d}\Omega \frac{B_{33}(-\Omega + 0\mathrm{i}, \eta_\alpha)}{\omega_0^2 - \Omega^2} \right\} \right]. \tag{4.4.8}$$

Equation (4.4.8) shows explicitly that $u_3(t, x)$ is constant over $S(t)$ and relates T_0 to u_0. The function $B_{33}(-\Omega + 0\mathrm{i}, \eta_\alpha)$ has a pole at $\Omega = c_\mathrm{R}(\eta)$ and the integral with respect to Ω in (4.4.8) is best determined numerically by deforming the contour in the lower half of the complex Ω-plane.

(*b*) *Dynamic indentation by a wedge*

A two-dimensional problem of some practical interest that has not been solved before involves the indentation of a half-space by a smooth rigid wedge moving uniformly. The corresponding three-dimensional problem of indentation by a cone was solved by Kostrov (1964*a*) and was briefly reviewed in § 4.3. For the wedge problem,

$$u_3(t, x) = \alpha |\nu_\lambda x_\lambda| - Ut \quad (x \in S(t)), \tag{4.4.9}$$

where the wdege has semiangle $\frac{1}{2}\pi-\alpha$ and moves into the half-space with speed U, making contact along the line $\nu_\lambda x_\lambda = 0$ at time $t=0$. The surface $S(t)$ here has the form

$$x_3 = 0, \quad |\nu_\lambda x_\lambda| \leqslant Vt, \tag{4.4.10}$$

where V is not known in advance, but is determined by the requirement that the pressure distribution beneath the wedge should tend to zero at the edge of the region of contact, $|\nu_\lambda x_\lambda| = Vt$.

For the boundary condition (4.4.9), $n=0$ and equation (4.2.7) implies

$$p^2G_3'(p+0\mathrm{i},\nu) - p^2G_3'(p-0\mathrm{i},\nu) = 0 \quad (|p|<V), \tag{4.4.11}$$

which defines a homogeneous Hilbert problem for the function $z^2G_3'(z,\nu)$. From (2.3.8), the function $z^2G_3'(z,\nu)$ is $O(1)$ as $|z|\to\infty$ and is $O((z\mp V)^{-1+\delta})$ as $z\to\pm V$, where $\delta>0$, since $T_3(p)\to 0$ as $p\to\pm V$. Hence,

$$z^2G_3'(z,\nu) = \frac{C+Dz}{(z^2-V^2)^{\frac{1}{2}}}, \tag{4.4.12}$$

where C and D are arbitrary. But it is also reasonable to require $T_3(p)$ to be integrable near $p=0$, and hence $C=0$. Thus,

$$G_3'(z,\nu) = \frac{D}{z(z^2-V^2)^{\frac{1}{2}}} \tag{4.4.13}$$

and
$$G_3(z,\nu) = \frac{\mathrm{i}D}{2V}\ln\left\{\frac{(z^2-V^2)^{\frac{1}{2}}+\mathrm{i}V}{(z^2-V^2)^{\frac{1}{2}}-\mathrm{i}V}\right\}. \tag{4.4.14}$$

Therefore, by the Plemelj formulae,

$$T_3(p) = \frac{\mathrm{i}D}{V}\ln\left\{\frac{V+(V^2-p^2)^{\frac{1}{2}}}{V-(V^2-p^2)^{\frac{1}{2}}}\right\} \equiv \frac{2\mathrm{i}D}{V}\ln\left\{\frac{V+(V^2-p^2)^{\frac{1}{2}}}{|p|}\right\}, \tag{4.4.15}$$

which has precisely the same form as for the conical indentor, equation (4.3.17). The total resultant pressure P per unit length of indentor is given by integrating (4.4.15); we have

$$P = -\int_{-Vt}^{Vt} T_3(t,x_\alpha)\,\mathrm{d}(\nu_\lambda x_\lambda) = -t\int_{-V}^{V} T_3(p)\,\mathrm{d}p = -2\pi\mathrm{i}Dt. \tag{4.4.16}$$

To fix the constant D (or $\dot{P}$) and V, equation (4.2.4) with $n=0$ must be integrated twice with respect to t. Integrating once, changing the variable of integration to $\Omega = \nu_\lambda x_\lambda/t$, gives

$$\partial_t u_3(t,x)|_{x_3=0} = 2(2\pi)^{-\frac{3}{2}}\dot{P}\int_{\nu_\lambda x_\lambda/t}^{\infty}\mathrm{d}\Omega\,\mathrm{Re}\left\{\frac{B_{33}(-\Omega+0\mathrm{i},\nu_\alpha)}{[(\Omega-0\mathrm{i})^2-V^2]^{\frac{1}{2}}}\right\}. \tag{4.4.17}$$

Now $B_{33}(-\Omega+0\mathrm{i},\nu_\alpha)$ is real, and hence the integrand is zero, if $|\Omega|<V$. Also, it is easily shown from equations (4.1.4) and (4.1.10) that the real part of $B_{33}(-\Omega+0\mathrm{i},\nu_\alpha)$ is an even function of Ω. while $[(\Omega-0\mathrm{i})^2-V^2]^{\frac{1}{2}}$ is odd if $|\Omega|>V$. Therefore, the upper limit ∞ can be replaced by $-\infty$, by deforming the contour of integration, and replacing Ω by $-\Omega$ then shows that the lower limit $\nu_\lambda x_\lambda/t$ may be replaced by $-\nu_\lambda x_\lambda/t$, or by its modulus, showing explicitly that the right side of (4.4.17) defines an even function of $(\nu_\lambda x_\lambda)$. Integrating (4.4.17) with respect to time, changing the order of integration and employing $|\nu_\lambda x_\lambda|/t$ as lower limit now gives

$$u_3(t,x)|_{x_3=0} = 2(2\pi)^{-\frac{3}{2}}\dot{P}\int_{|\nu_\lambda x_\lambda|/t}^{c_m(\nu)}\mathrm{d}\Omega\left(t-\frac{|\nu_\lambda x_\lambda|}{\Omega}\right)\mathrm{Re}\left\{\frac{B_{33}(-\Omega+0\mathrm{i},\nu_\alpha)}{[(\Omega-0\mathrm{i})^2-V^2]^{\frac{1}{2}}}\right\}, \tag{4.4.18}$$

where the upper limit of integration has been replaced by $c_m(\nu)$, the greatest speed of propagation of plane waves in the direction ν, since $B_{33}(-\Omega+0\mathrm{i},\nu)$ is imaginary for $\Omega > c_m(\nu)$. Hence, finally, when $|\nu_\lambda x_\lambda| < Vt$,

$$u_3(t,x)|_{x_3=0} = \alpha|\nu_\lambda x_\lambda| - Ut = 2(2\pi)^{-\frac{3}{2}}\dot{P}\,\mathrm{Re}\left\{\int_V^{c_m(\nu)} \mathrm{d}\Omega\left(t-\frac{|\nu_\lambda x_\lambda|}{\Omega}\right)\frac{B_{33}(-\Omega+0\mathrm{i},\nu_\alpha)}{(\Omega^2-V^2)^{\frac{1}{2}}}\right\}, \quad (4.4.19)$$

which determines $\dot{P}$ and V in terms of α and U. Again, the integral in (4.4.19) is best evaluated by deforming the contour to avoid passing close to the Rayleigh pole of $B_{33}(-\Omega+0\mathrm{i},\nu_\alpha)$.

4.5. *Crack problems for an infinite homogeneous medium*

Suppose that an infinite homogeneous medium is loaded in a way that would produce a stress field $\sigma^0_{ij}(t,x)$, which is homogeneous of degree n in (t,x), if no crack were present. Now suppose that a crack appears at the origin at time $t=0$, and subsequently expands to cover the surface $S(t)$:

$$x_3 = 0, \quad s_1^2x_1^2+s_2^2x_2^2 \leqslant t^2. \quad (4.1.2)$$

The presence of the crack induces additional stress and displacement fields $\sigma_{ij}(t,x)$, $u_i(t,x)$, which are homogeneous of degrees n and $n+1$ respectively. These additional fields must satisfy the equation of motion (2.1.3), the initial conditions (2.1.4) and the mixed boundary conditions

$$\left.\begin{aligned}\sigma_{i3}(t,x) = c_{i3kl}u_{k,l}(t,x) &= -\tau^0_i(t,x_\alpha) \quad (x\in S(t)),\\ [u(t,x)]^{x_3=+0}_{x_3=-0} &= 0 \quad (x\notin S(t)),\\ u(t,x)&\to 0 \quad (|x|\to\infty),\end{aligned}\right\} \quad (4.5.1)$$

where
$$\tau^0_i(t,x_\alpha) = \sigma^0_{i3}(t,x_1,x_2,0). \quad (4.5.2)$$

The first of equations (4.5.1) ensures that the total traction $\sigma_{i3}+\sigma^0_{i3}$ is zero on the crack faces, while the second states that the displacement is continuous except across the crack. If the discontinuity in $u(t,x)$ across the crack were known, say

$$[u(t,x)] = b(t,x_\alpha), \quad (4.5.3)$$

where $b(t,x_\alpha)$ is zero outside $S(t)$, the problem would be reduced to a particular case of that of § 3.1, with the two half-spaces identical. In particular, equation (3.1.13) would give

$$\partial_t^{(n+1)}T(t,x_\alpha) = \frac{(2\pi)^{-\frac{5}{2}}}{t}\oint_{|\eta|=1}\mathrm{d}s\left(-\frac{\eta_\alpha x_\alpha}{t}\right)^{n+1}\mathrm{Re}\left\{A^{-1}\left(-\frac{\eta_\alpha x_\alpha}{t}+0\mathrm{i},\eta_\alpha\right)F^{(n+3)}\left(\frac{\eta_\alpha x_\alpha}{t}-0\mathrm{i},\eta\right)\right\}, \quad (4.5.4)$$

the insertion of the 'real part' sign being justified by reasoning like that following equation (2.1.36). Now

$$\sum_{N=1}^{3} B^{-N}(\Omega+0\mathrm{i},-\eta_\alpha) = -\sum_{N=4}^{6} B^{+N}(\Omega+0\mathrm{i},\eta_\alpha), \quad (4.5.5)$$

since
$$\xi_3^N(\Omega+0\mathrm{i},-\eta_\alpha) = -\xi_3^M(\Omega+0\mathrm{i},\eta_\alpha)$$

for some M in the range (4, 5, 6) if N is in the range (1, 2, 3). Therefore, $A(\Omega+0\mathrm{i},\eta_\alpha)$ is an even function of η and equation (4.1.4) now implies that $A(\Omega+0\mathrm{i},\eta_\alpha)$ is symmetric, and hence real when Ω is subsonic. Therefore, when $x\in S(t)$, $A^{-1}(-\eta_\alpha x_\alpha/t+0\mathrm{i},\eta_\alpha)$ is real and equation (4.5.4) simplifies upon use of the Plemelj formulae to

$$\partial_t^{(n+1)}T(t,x_\alpha) = -\frac{(2\pi)^{-\frac{5}{2}}}{2t}\oint_{|\eta|=1}\mathrm{d}s\left(-\frac{\eta_\alpha x_\alpha}{t}\right)^{n+1}A^{-1}\left(-\frac{\eta_\alpha x_\alpha}{t}+0\mathrm{i},\eta_\alpha\right)\breve{b}^{(n+3)}\left(\frac{\eta_\alpha x_\alpha}{t},\eta_\alpha\right), \quad (4.5.6)$$

where
$$\breve{b}^{(n+3)}(\Omega, \eta_\alpha) = \partial_\Omega^{(n+3)} \breve{b}(\Omega, \eta_\alpha) \tag{4.5.7}$$

and $\breve{b}(\Omega, \eta_\alpha)$ is the Radon transform of $b(t, x_\alpha)$. Equation (4.5.6) is an integral equation for $\breve{b}^{(n+3)}(\Omega, \eta_\alpha)$, and hence implicitly for $b(t, x_\alpha)$. Its similarity to (4.1.14) may be noted, and a result corresponding to that summarized in equations (4.1.24) and (4.1.25) may be proved for the present problem. If

$$b(t, x_\alpha) = P_n(t, x_\alpha)\,(t^2 - s_1^2 x_1^2 - s_2^2 x_2^2)^{\frac{1}{2}}\, H(t^2 - s_1^2 x_1^2 - s_2^2 x_2^2), \tag{4.5.8}$$

where P_n is a homogeneous polynomial of degree n, then $b(t, x_\alpha)$ is bounded and its Radon transform $\breve{b}(\Omega, \eta_\alpha)$ is a polynomial of degree $n+2$ in Ω. The right side of equation (4.5.6) is therefore zero, and the following result is obtained immediately:

The relative displacement (4.5.8) across $S(t)$ is produced by a traction across $S(t)$ of the form

$$T(t, x_\alpha) = Q_n(t, x_\alpha), \tag{4.5.9}$$

where Q_n is a polynomial of degree n. The restriction to homogeneous polynomials P_n and Q_n may be dropped, since any polynomial is a sum of homogeneous polynomials.

The coefficients in P_n may be related to those in Q_n by integrating equation (4.5.4) with respect to time. Details will not be pursued at this point but some examples will be discussed in § 4.7.

The above result is the most general that is known for truly three-dimensional situations (problems with axial symmetry are discussed in § 4.6). Burridge & Willis (1969) have previously derived the result for the case $n = 0$ (constant loading at infinity), generalizing the results of Kostrov (1964*b, c*) for a circular crack in an isotropic medium, but the method that was employed was less systematic than the one presented above.

A corresponding treatment may be given for the two-dimensional problem, in which $S(t)$ is the surface

$$x_3 = 0, \quad |\nu_\lambda x_\lambda| \leqslant Vt \tag{4.4.10}$$

and
$$T(t, x_\alpha) = t^n T(p), \tag{2.3.2}$$

where V is subsonic, $p = \nu_\lambda x_\lambda / t$ and $T(p)$ is given for $|p| < V$. In this case, when $x \in S(t)$,

$$\partial_t^{(n+1)} T(t, x_\alpha) = \frac{1}{t} T^{(n+1)}(p), \quad \text{say}, \tag{4.5.10}$$

and equation (3.2.7) implies

$$T^{(n+1)}(p) = \frac{(2\pi)^{-\frac{7}{2}}(-1)^n}{\mathrm{i}} p^{n+1} A^{-1}(-p+0\mathrm{i}, \nu_\alpha)\,\{G^{(n+2)}(p-0\mathrm{i}, \nu) + G^{(n+2)}(p+0\mathrm{i}, \nu)\}, \tag{4.5.11}$$

which is a Hilbert problem for $z^{n+1}G^{(n+2)}(z, \nu)$, to be solved subject to the conditions

$$\left.\begin{aligned} z^{n+1}G^{(n+2)}(z, \nu) &= O(z^{-2}) \quad (|z| \to \infty), \\ z^{n+1}G^{(n+2)}(z, \nu) &= O[(z \mp V)^{-n-2+\delta}] \quad (z \to \pm V) \end{aligned}\right\} \tag{4.5.12}$$

for some $\delta > 0$, since $b(t, x_\alpha)$ tends to zero as $|\nu_\lambda x_\lambda| \to Vt$. The most general admissible solution of (4.5.11) is therefore

$$z^{n+1}G^{(n+2)}(z, \nu) = \mathrm{i}(2\pi)^{\frac{5}{2}}(-1)^n (z^2 - V^2)^{-\frac{1}{2}} \int_{-V}^{V} \frac{A(-p+0\mathrm{i}, \nu_\alpha)\, T^{(n+1)}(p)\,(V^2 - p^2)^{\frac{1}{2}}\,\mathrm{d}p}{p - z}$$
$$+ P_{2n+1}(z)\,(z^2 - V^2)^{-n-\frac{3}{2}}, \tag{4.5.13}$$

where P_{2n+1} is a polynomial of degree $2n+1$ whose coefficients must be chosen so that $G^{(n+2)}(z, \nu)$ is finite at $z = 0$ and so that $G^{(n+2)}(z, \nu)$ leads to the desired $T(t, x_\alpha)$. Again, an example will be given in § 4.7.

If $T(t, x_\alpha)$ is a homogeneous polynomial of degree n in $(t, \nu_\lambda x_\lambda)$, then $T^{(n+1)}(p) = 0$ and

$$G^{(n+2)}(z, \nu) = P_n(z)\,(z^2 - V^2)^{-n-\frac{3}{2}}, \tag{4.5.14}$$

where P_n is a polynomial of degree n. Equation (4.5.14) may be integrated to give a result of the form

$$G(z, \nu) = Q_n(z)\,(z^2 - V^2)^{\frac{1}{2}} + R_{n+1}(z), \tag{4.5.15}$$

where Q_n and R_{n+1} are polynomials of the degrees indicated. The Plemelj formulae now give

$$b(p) = 2\mathrm{i} Q_n(p)\,(V^2 - p^2)^{\frac{1}{2}}, \tag{4.5.16}$$

so that the traction $T(t, x_\alpha)$ produces the relative displacement $b(t, x_\alpha) = t^{n+1} b(p)$, which is similar in form to (4.5.8). The restriction to homogeneous polynomials may once more be relaxed, to yield the result:

The traction
$$T(t, x_\alpha) = P_n(t, \nu_\lambda x_\lambda) \tag{4.5.17}$$
over $S(t)$ produces the relative displacement

$$b(t, x_\alpha) = Q_n(t, \nu_\lambda x_\lambda)\,(V^2 t^2 - (\nu_\lambda x_\lambda)^2)^{\frac{1}{2}} \tag{4.5.18}$$

over $S(t)$, where P_n and Q_n are polynomials of degree n.

Two-dimensional problems have been previously treated less generally by Broberg (1960), Craggs (1963 *b*) and Atkinson (1967), all for a crack in an isotropic body, and by Atkinson (1965) for a crack in an orthotropic body. All of these authors discussed only the case $n = 0$, for purely tensile loading.

4.6. *Axisymmetric crack problems*

If the medium is isotropic, or transversely isotropic about the x_3-axis, then $B(\Omega, \eta_\alpha)$ has the form given in equation (2.6.13) and $A(\Omega + 0\mathrm{i}, \eta_\alpha)$ is twice its even part. Thus,

$$A(\Omega + 0\mathrm{i}, \eta_\alpha) = 2\begin{bmatrix} b + c\eta_2^2, & -c\eta_1\eta_2, & 0 \\ -c\eta_1\eta_2, & b + c\eta_1^2, & 0 \\ 0, & 0, & e \end{bmatrix}, \tag{4.6.1}$$

in which b, c and e are functions of Ω that are real and even when Ω is subsonic.

Now let $S(t)$ be the circle

$$x_3 = 0, \quad r \leqslant Vt, \tag{4.3.1}$$

where $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$, and let

$$T(t, x_\alpha) = t^n T_r(q)\begin{bmatrix} x_1/r \\ x_2/r \\ 0 \end{bmatrix} + t^n T_\theta(q)\begin{bmatrix} -x_2/r \\ x_1/r \\ 0 \end{bmatrix} + t^n T_3(q)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{4.6.2}$$

where $q = r/t$. Correspondingly, the relative displacement $b(t, x_\alpha)$ has the form

$$b(t, x_\alpha) = t^{n+1} b_r(q)\begin{bmatrix} x_1/r \\ x_2/r \\ 0 \end{bmatrix} + t^{n+1} b_\theta(q)\begin{bmatrix} -x_2/r \\ x_1/r \\ 0 \end{bmatrix} + t^{n+1} b_3(q)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.6.3}$$

The Radon transform of $b(t, x_\alpha)$ is now easily calculated as

$$\check{b}(\Omega, \eta_\alpha) = \check{b}_r(\Omega)\begin{bmatrix} \eta_1 \\ \eta_2 \\ 0 \end{bmatrix} + \check{b}_\theta(\Omega)\begin{bmatrix} -\eta_2 \\ \eta_1 \\ 0 \end{bmatrix} + \check{b}_3(\Omega)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{4.6.4}$$

where
$$\left.\begin{aligned}\breve{b}_{r,\theta}(\Omega) &= 2\Omega\int_{|\Omega|}^{V} b_{r,\theta}(q)\,\frac{\mathrm{d}q}{(q^2-\Omega^2)^{\frac{1}{2}}},\\ \breve{b}_3(\Omega) &= 2\int_{|\Omega|}^{V} b_3(q)\,\frac{q\,\mathrm{d}q}{(q^2-\Omega^2)^{\frac{1}{2}}},\end{aligned}\right\} \tag{4.6.5}$$

with inverses
$$\left.\begin{aligned}b_{r,\theta}(q) &= -\frac{q}{\pi}\int_q^V \frac{\mathrm{d}(\breve{b}_{r,\theta}(\Omega)/\Omega)}{\mathrm{d}\Omega}\,\frac{\mathrm{d}\Omega}{(\Omega^2-q^2)^{\frac{1}{2}}},\\ b_3(q) &= -\frac{1}{\pi}\int_q^V \frac{\mathrm{d}\breve{b}_3(\Omega)}{\mathrm{d}\Omega}\,\frac{\mathrm{d}\Omega}{(\Omega^2-q^2)^{\frac{1}{2}}}.\end{aligned}\right\} \tag{4.6.6}$$

A simple calculation now shows that
$$A^{-1}(\Omega+0\mathrm{i},\eta_\alpha)\,\breve{b}^{(n+3)}(-\Omega,\eta_\alpha) = \frac{\breve{b}_r^{(n+3)}(-\Omega)}{2b}\begin{bmatrix}\eta_1\\ \eta_2\\ 0\end{bmatrix} + \frac{\breve{b}_\theta^{(n+3)}(-\Omega)}{2(b+c)}\begin{bmatrix}-\eta_2\\ \eta_1\\ 0\end{bmatrix} + \frac{\breve{b}_3^{(n+3)}(-\Omega)}{2e}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}, \tag{4.6.7}$$

and equation (4.5.6) reduces, upon introducing $\Omega = \eta_\alpha x_\alpha/t$ as a variable of integration, to a form which is equivalent to
$$\left.\begin{aligned}T_r^{(n+1)}(q) &= \frac{(-1)^n(2\pi)^{-\frac{5}{2}}}{q}\int_0^q \frac{\Omega^{n+2}}{b}\,\breve{b}_r^{(n+3)}(\Omega)\,\frac{\mathrm{d}\Omega}{(q^2-\Omega^2)^{\frac{1}{2}}},\\ T_\theta^{(n+1)}(q) &= \frac{(-1)^n(2\pi)^{-\frac{5}{2}}}{q}\int_0^q \frac{\Omega^{n+2}}{b+c}\,\breve{b}_\theta^{(n+3)}(\Omega)\,\frac{\mathrm{d}\Omega}{(q^2-\Omega^2)^{\frac{1}{2}}},\\ T_3^{(n+1)}(q) &= (-1)^n(2\pi)^{-\frac{5}{2}}\int_0^q \frac{\Omega^{n+1}}{e}\,\breve{b}_3^{(n+3)}(\Omega)\,\frac{\mathrm{d}\Omega}{(q^2-\Omega^2)^{\frac{1}{2}}},\end{aligned}\right\} \tag{4.6.8}$$

where $T_r^{(n+1)}(q)$ is the radial component of $t\,\partial_t^{(n+1)}\,T(t,x_\alpha)$, and so on. Equations (4.6.8) are three uncoupled equations of Abel type, and a complete solution of the problem now follows in the same way as that for the dual problem discussed in § 4.3. It may be noted that, for an isotropic medium,
$$(b+c) = (2\pi)^{-\frac{3}{2}}(\Omega^2/\beta^2-1)^{-\frac{1}{2}}/(\mathrm{i}\mu),$$

so that the second of equations (4.6.8), which governs the torsional component, involves only the shear modulus μ.

Axisymmetric problems have not been treated before at the present level of generality. The only available formulation of a problem for non-uniform loading is that of Atkinson & Innes (1969), who briefly sketched a method of solving problems for non-uniform tensile loading, with $n = 0$, for an isotropic half-space. They constructed an equation of Abel type by forming a superposition of a rather complicated one-parameter family of solutions that had been found by Webb & Atkinson (1969).

4.7. *Examples*

(*a*) *Constant loading at infinity* ($n = 0$)

As a first example, consider the particular case of the three-dimensional problem discussed in § 4.5 for which $n = 0$. Then τ^0 is constant over $S(t)$,
$$b(t,x_\alpha) = b(t^2-s_1^2x_1^2-s_2^2x_2^2)^{\frac{1}{2}}\,H(t^2-s_1^2x_1^2-s_2^2x_2^2), \tag{4.7.1}$$

where b is a constant vector, and use of (4.1.19) shows that
$$\breve{b}(\Omega,\eta_\alpha) = \frac{b\pi}{2s_1s_2\omega_0}\left(1-\frac{\Omega^2}{\omega_0^2}\right)H\left(1-\frac{\Omega^2}{\omega_0^2}\right), \tag{4.7.2}$$

where $\omega_0(\eta)$ is given by equation (4.4.4). A straightforward calculation now gives

$$F^{(3)}(z,\eta) = \frac{2b}{\mathrm{i}s_1 s_2(\omega_0^2 - z^2)^2}, \tag{4.7.3}$$

so that, from equation (4.5.4),

$$\partial_t T(t,x_\alpha) = \frac{2(2\pi)^{-\frac{5}{2}}}{s_1 s_2 t}\oint_{|\eta|=1} \mathrm{d}s\left(\frac{\eta_\alpha x_\alpha}{t}\right)\mathrm{Re}\left\{\mathrm{i}A^{-1}\left(-\frac{\eta_\alpha x_\alpha}{t}+0\mathrm{i},\eta_\alpha\right)b\left[\omega_0^2-\left(\frac{\eta_\alpha x_\alpha}{t}-0\mathrm{i}\right)^2\right]^{-2}\right\}. \tag{4.7.4}$$

Hence, by integrating with respect to time, putting $\Omega = \eta_\alpha x_\alpha/t$,

$$T(t,x_\alpha) = \frac{2(2\pi)^{-\frac{5}{2}}}{s_1 s_2}\oint_{|\eta|=1} \mathrm{d}s\,\mathrm{Re}\left\{\mathrm{i}\int_{\eta_\alpha x_\alpha/t}^{\infty}\mathrm{d}\Omega\,A^{-1}(-\Omega+0\mathrm{i},\eta_\alpha)\,b[\omega_0^2-(\Omega-0\mathrm{i})^2]^{-2}\right\}, \tag{4.7.5}$$

which is independent of x when $x\in S(t)$. The vector b is therefore determined from the equation

$$\frac{2(2\pi)^{-\frac{5}{2}}}{s_1 s_2}\oint_{|\eta|=1} \mathrm{d}s\,\mathrm{Re}\left\{\mathrm{i}\int_0^{\infty}\mathrm{d}\Omega\,A^{-1}(-\Omega+0\mathrm{i},\eta_\alpha)\,b[\omega_0^2-(\Omega-0\mathrm{i})^2]^{-2}\right\} = -\tau^0. \tag{4.7.6}$$

The elastic arrivals are then given by equation (2.2.15), (2.2.19) or (2.2.26), with $n = 0$ and

$$\tilde{T}(\omega,\xi_\alpha) = -\frac{A^{-1}(\omega,\xi_\alpha)\,b}{2\pi^2 s_1 s_2[(\omega_0(\xi))^2-\omega^2]^2}. \tag{4.7.7}$$

A further singularity of some interest is that in $T(t,x_\alpha)$ when x lies just outside $S(t)$. This is probably easiest to obtain by first transforming the integral in (4.7.4) to an integral around the 'slowness curve' C of the crack, defined by the equation

$$\omega_0(\zeta) = 1. \tag{4.7.8}$$

Use of the homogeneity of the integrand then gives

$$\partial_t T(t,x_\alpha) = -\frac{2(2\pi)^{-\frac{5}{2}}}{s_1 s_2 t}\oint_C \frac{\mathrm{d}s}{|\nabla\omega_0|}\left(\frac{\zeta_\alpha x_\alpha}{t}\right)\mathrm{Re}\left\{\mathrm{i}A^{-1}\left(-\frac{\zeta_\alpha x_\alpha}{t}+0\mathrm{i},\zeta_\alpha\right)b\left[1-\left(\frac{\zeta_\alpha x_\alpha}{t}+0\mathrm{i}\right)^2\right]^{-2}\right\}, \tag{4.7.9}$$

which shows that $\partial_t T(t,x_\alpha)$ is singular at the time $t = t_0 = (s_1^2x_1^2+s_2^2x_2^2)^{\frac{1}{2}}$, when the line

$$t+\zeta_\alpha x_\alpha = 0 \tag{4.7.10}$$

touches the curve C, at the point ζ^0, say. Therefore, as $t\to t_0$,

$$\partial_t T(t,x_\alpha) \sim -\frac{4(2\pi)^{-\frac{5}{2}}}{s_1 s_2 t_0|\nabla\omega_0|}\mathrm{Re}\left\{\mathrm{i}A^{-1}(1+0\mathrm{i},\zeta_\alpha^0)\,b\int_{\mathscr{N}(\zeta^0)}\mathrm{d}s\left[1-\left(\frac{\zeta_\alpha x_\alpha}{t}+0\mathrm{i}\right)^2\right]^{-2}\right\}, \tag{4.7.11}$$

where $\mathscr{N}(\zeta^0)$ is a small arc of C containing the point ζ^0, and use has been made of the fact that the singularities at ζ^0 and $-\zeta^0$ contribute equally. The integrand of equation (4.7.11) may now be replaced asymptotically by $-4t_0\partial_t[1+\zeta_\alpha x_\alpha/t+0\mathrm{i}]^{-1}$, after which an integration with respect to time yields

$$T(t,x_\alpha) \sim \frac{(2\pi)^{-\frac{5}{2}}}{s_1 s_2|\nabla\omega_0|}\mathrm{Re}\left\{\mathrm{i}A^{-1}(1+0\mathrm{i},\zeta_\alpha^0)\,b\int_{\mathscr{N}(\zeta^0)}\mathrm{d}s\left[1+\frac{\zeta_\alpha x_\alpha}{t}+0\mathrm{i}\right]^{-1}\right\}. \tag{4.7.12}$$

The integral in (4.7.12) can be evaluated from the results of § 2.2 by setting

$$(\zeta_\alpha^0-\zeta_\alpha)\,x_\alpha = \tfrac{1}{2}a|x|\,s^2, \tag{4.7.13}$$

which is a good approximation in $\mathcal{N}(\zeta^0)$, so that the integrand in (4.7.12) becomes, asymptotically, $t_0[t-t_0+0\mathrm{i}]^{-1}$. Use of the easily derived relations

$$\zeta_1^0 = -s_1^2 x_1/t, \quad \zeta_2^0 = -s_2^2 x_2/t_0, \tag{4.7.14}$$

$$a = -\frac{1}{s_1^2 s_2^2}\left(\frac{t_0}{|x|}\right)^3, \quad \nabla\omega_0 = x/t_0, \tag{4.7.15}$$

and equation (2.2.40) now gives

$$T(t, x_\alpha) \sim \frac{1}{4\pi^{\frac{3}{2}}} A^{-1}(1+0\mathrm{i}, \zeta_\alpha^0)\, b\left(\frac{t_0}{t_0-t}\right)^{\frac{1}{2}} H(t_0-t) \tag{4.7.16}$$

as $t \to t_0$, displaying the expected square root singularity.

The results of this section have previously been derived in a rather different form by Burridge & Willis (1969).

(*b*) *Linearly varying loads* ($n = 1$)

Suppose now

$$\tau^0(t, x_\alpha) = \tau_1 x_1 + \tau_2 x_2, \tag{4.7.17}$$

where τ_1 and τ_2 are constant vectors. Then from § 4.5,

$$b(t, x_\alpha) = (b_1 x_1 + b_2 x_2)\,(t^2 - s_1^2 x_1^2 - s_2^2 x_2^2)^{\frac{1}{2}} H(t^2 - s_1^2 x_1^2 - s_2^2 x_2^2), \tag{4.7.18}$$

where b_1 and b_2 are constant vectors (a term proportional to t in the first bracket should strictly be added at this point but the work to follow will show that it is not needed). It can now be shown, using equation (4.1.19), that

$$\check{b}(\Omega, \eta_\alpha) = \frac{2\pi}{\mathrm{i}s_1 s_2 \omega_0^2}\left(\frac{b_1\eta_1}{s_1^2} + \frac{b_2\eta_2}{s_2^2}\right)\frac{\Omega}{\omega_0}\left[1-\left(\frac{\Omega}{\omega_0}\right)^2\right] \tag{4.7.19}$$

and, correspondingly, that

$$F^{(4)}(z, \eta) = \frac{32}{\mathrm{i}s_1 s_2}\left(\frac{b_1\eta_1}{s_1^2} + \frac{b_2\eta_2}{s_2^2}\right)(\omega_0^2 - z^2)^{-3}. \tag{4.7.20}$$

Therefore, from (4.5.4),

$$\partial_t^2 T(t, x_\alpha) = -\frac{32(2\pi)^{-\frac{5}{2}}}{s_1 s_2 t}\oint_{|\eta|=1} \mathrm{d}s\left(\frac{\eta_\alpha x_\alpha}{t}\right)^2 \times \mathrm{Re}\left\{\mathrm{i}A^{-1}\left(-\frac{\eta_\alpha x_\alpha}{t}+0\mathrm{i}, \eta_\alpha\right)\left(\frac{b_1\eta_1}{s_1^2}+\frac{b_2\eta_2}{s_2^2}\right)\left[\omega_0^2 - \left(\frac{\eta_\alpha x_\alpha}{t} - 0\mathrm{i}\right)^2\right]^{-3}\right\}. \tag{4.7.21}$$

Integrating (4.7.21) once with respect to time yields

$$\partial_t T(t, x_\alpha) = -\frac{32(2\pi)^{-\frac{5}{2}}}{s_1 s_2}\oint_{|\eta|=1} \mathrm{d}s\,\mathrm{Re}\left\{\mathrm{i}\int_{\eta_\alpha x_\alpha/t}^{\infty} \Omega\,\mathrm{d}\Omega\, A^{-1}(-\Omega+0\mathrm{i}, \eta_\alpha) \times\left(\frac{b_1\eta_1}{s_1^2}+\frac{b_2\eta_2}{s_2^2}\right)[\omega_0^2 - (\Omega - 0\mathrm{i})^2]^{-3}\right\}, \tag{4.7.22}$$

which may be integrated again, changing the orders of integration, to give

$$T(t, x_\alpha) = -\frac{32(2\pi)^{-\frac{5}{2}}}{s_1 s_2}\oint_{|\eta|=1} \mathrm{d}s\,\mathrm{Re}\left\{\mathrm{i}\int_{\eta_\alpha x_\alpha/t}^{\infty} \mathrm{d}\Omega(\Omega t - \eta_\alpha x_\alpha)\, A^{-1}(-\Omega+0\mathrm{i}, \eta_\alpha) \times\left(\frac{b_1\eta_1}{s_1^2}+\frac{b_2\eta_2}{s_2^2}\right)[\omega_0^2 - (\Omega - 0\mathrm{i})^2]^{-3}\right\}; \tag{4.7.23}$$

this form appears immediately if $\eta_\alpha x_\alpha/t > 0$, and for $\eta_\alpha x_\alpha/t < 0$ by exploiting the fact that the integration with respect to Ω in (4.7.29) may be performed along any path from $\eta_\alpha x_\alpha/t$ to ∞ so that the upper limit ∞ can be replaced by $-\infty$ whenever this is convenient. When $x \in S(t)$, the integral with respect to Ω is independent of the lower limit $\eta_\alpha x_\alpha/t$, which may therefore be set equal to zero. The even property of A^{-1} as a function of η now shows that the term Ωt contributes nothing and so, when $x \in S(t)$,

$$T(t, x_\alpha) = \frac{32(2\pi)^{-\frac{5}{2}}}{s_1 s_2} \oint_{|\eta|=1} \mathrm{d}s \,\mathrm{Re}\left\{\mathrm{i}\int_0^\infty \mathrm{d}\Omega\, \eta_\alpha x_\alpha A^{-1}(-\Omega+0\mathrm{i}, \eta_\alpha) \times \left(\frac{b_1\eta_1}{s_1^2}+\frac{b_2\eta_2}{s_2^2}\right)[\omega_0^2-(\Omega-0\mathrm{i})^2]^{-3}\right\}, \tag{4.7.24}$$

which, when set equal to $-(\tau_1 x_1 + \tau_2 x_2)$, provides equations for the vectors b_1, b_2.

As in example (*a*), the body-wave arrivals are given by equations (2.2.15), (2.2.19) or (2.2.26), with $n = 1$ and

$$\tilde{T}(\omega, \xi_\alpha) = \frac{8\mathrm{i}A^{-1}(\omega, \xi_\alpha)}{\pi^2 s_1 s_2}\left(\frac{b_1\xi_1}{s_1^2}+\frac{b_2\xi_2}{s_2^2}\right)[(\omega_0(\xi))^2-\omega^2]^{-3}. \tag{4.7.25}$$

The singularity in $T(t, x_\alpha)$ for x close to $S(t)$ may be found by the method that was employed in the preceding example. Similar manipulations starting from (4.7.21) show that

$$\partial_t^2 T(t, x_\alpha) \sim -\frac{4(2\pi)^{-\frac{5}{2}}}{s_1 s_2}\frac{t_0}{|\nabla\omega_0|}\partial_t^2\,\mathrm{Re}\left\{\mathrm{i}A^{-1}(1+0\mathrm{i}, \zeta_\alpha^0)\left(\frac{b_1\zeta_1^0}{s_1^2}+\frac{b_2\zeta_2^0}{s_2^2}\right)\int_{\mathcal{N}(\xi^0)}\mathrm{d}s\left[1+\frac{\zeta_\alpha x_\alpha}{t}+0\mathrm{i}\right]^{-1}\right\} \tag{4.7.26}$$

as $t \to t_0$. Comparison with (4.7.12) now gives immediately

$$T(t, x_\alpha) \sim \pi^{-\frac{3}{2}}A^{-1}(1+0\mathrm{i}, \zeta_\alpha^0)\,(b_1 x_1 + b_2 x_2)\left(\frac{t_0}{t_0-t}\right)^{\frac{1}{2}} H(t_0 - t). \tag{4.7.27}$$

(*c*) *The self-similar relaxed crack* ($n = 0$)

An example will now be considered for which the extension of 'Galin's theorem' embodied in equations (4.5.17), (4.5.18) is not applicable. Consider a crack occupying the region $S(t)$:

$$x_3 = 0, \quad |x_1| < Ut, \tag{4.7.28}$$

in a medium for which the plane $x_3 = 0$ is a plane of symmetry, so that $A_{13} = A_{23} = 0$. The crack expands in response to a uniform tension $\sigma_{33}^0 = \tau^0$ applied at infinity, and yielding takes place in the region $Ut \leqslant |x_1| \leqslant Vt$ in the plane of the crack, so that the stress σ_{33} is everywhere finite. In the zone of yielding, σ_{33} takes the constant value Y, and the speed V with which this zone expands is determined by the condition that σ_{33} remains bounded near $x_1 = \pm Vt$. This simple model of plastic yielding ahead of a crack was first proposed by Dugdale (1960) and its extension to the dynamic situation described here was carried out by Atkinson (1967). The problem is reconsidered here both as a simple application of the results of § 4.5 and because the present approach shows up immediately a simple feature of the problem that was not at all apparent in Atkinson's treatment.

If u_i and σ_{ij} are defined on the additional displacements and stresses that are induced by the crack, the boundary conditions corresponding to (4.5.1) that define them are

$$\left.\begin{aligned} \sigma_{i3}(t,x) &= \{YH(|x_1|-Ut)-\tau^0\}\,\delta_{i3} \quad (x_3 = \pm 0,\ |x_1| < Vt), \\ [u(t,x)] &= 0 \quad (|x_1| > Vt), \\ u(t,x) &\to 0 \quad (|x| \to \infty). \end{aligned}\right\} \tag{4.7.29}$$

Thus, although physically the crack occupies only the region $|x_1| < Ut$ and is loaded uniformly, the mathematical problem that is to be solved is equivalent to one for a crack over the region $|x_1| < Vt$, which is loaded in the way specified above.

The solution of the problem follows directly from equation (4.5.13) with $n = 0$ and

$$T_i^{(1)}(p) = -YU\{\delta(x_1 - Ut) + \delta(x_1 + Ut)\}\,\delta_{i3}. \tag{4.7.30}$$

Thus, since $A_{13} = A_{23} = 0$, the only non-zero component of $G^{(2)}(z)$ is $G_3^{(2)}(z)$ and this is given as

$$G_3^{(2)}(z) = \frac{2(2\pi)^{\frac{5}{2}}\,YUA_{33}(-U+0\mathrm{i})\,(V^2-U^2)^{\frac{1}{2}}\,\mathrm{i}}{(z^2-V^2)^{\frac{1}{2}}\,(U^2-z^2)} + \frac{P_1}{(z^2-V^2)^{\frac{3}{2}}}, \tag{4.7.31}$$

where P_1 is a constant. But $G_3^{(1)}(z)$ must be finite at $z = \pm V$, since the stress σ_{33} is bounded at $x_1 = \pm Vt$. Hence,

$$P_1 = 0. \tag{4.7.32}$$

It may be noted that the dynamic effects appear in a very simple way in (4.7.31), through the term $A_{33}(-U+0\mathrm{i})$, which was picked out by the delta functions in (4.7.30). Thus, the relative displacement $b_3(t, x_1) = tb_3(1,p)$, which is obtained from the Plemelj formula

$$b_3(1,p) = G_3(p+0\mathrm{i}) - G_3(p-0\mathrm{i}) \tag{4.7.33}$$

has the same form independently of the speed U, though its amplitude is governed by the factor $A_{33}(-U+0\mathrm{i})$. The speed V of the zone yielding is now fixed by substituting $G_3^{(2)}(z)$ into (3.2.7) and integrating with respect to time to give $T(t, x_1)$. Identifying $T_3(t, 0)$ with $-\tau^0$ then gives

$$\tau^0 = \frac{YUA_{33}(-U+0\mathrm{i})\,(V^2-U^2)^{\frac{1}{2}}}{\pi}\,\mathrm{Re}\left\{\int_{0\mathrm{i}}^{\infty+0\mathrm{i}} \frac{\mathrm{d}\Omega}{(\Omega^2-V^2)^{\frac{1}{2}}\,(U^2-\Omega^2)\,A_{33}(\Omega)}\right\} \tag{4.7.34}$$

as an equation for V in terms of τ^0. Numerical results relating τ^0, U and V have been given by Atkinson (1970), but the 'universal' character of the relative displacement $b_3(t,x_1)$ has not been noted before. Equation (4.7.31) can be integrated to give

$$\begin{aligned} G_3(z) = (2\pi)^{\frac{5}{2}}\,YA_{33}(-U+0\mathrm{i})\Big\{&(z-U)\ln\left(\frac{V^2-Uz-\mathrm{i}(V^2-U^2)^{\frac{1}{2}}(z^2-V^2)^{\frac{1}{2}}}{z-U}\right) \\ &-(z+U)\ln\left(\frac{V^2+Uz-1(V^2-U^2)^{\frac{1}{2}}(z^2-V^2)^{\frac{1}{2}}}{z+U}\right)+k_1z+k_2\Big\}, \end{aligned} \tag{4.7.35}$$

where k_1 and k_2 take values so that $G_3(z)$ and $G_3'(z)$ tend to zero as z tends to infinity. The Plemelj formula (4.7.33) now gives

$$\begin{aligned} b_3(1,p) = 2(2\pi)^{\frac{5}{2}}\,YA_{33}(-U+0\mathrm{i})\Big\{&(p-U)\ln\left(\frac{V^2-Up+(V^2-U^2)^{\frac{1}{2}}(V^2-p^2)^{\frac{1}{2}}}{V|p-U|}\right) \\ &-(p+U)\ln\left(\frac{V^2+Up+(V^2-U^2)^{\frac{1}{2}}(V^2-p^2)^{\frac{1}{2}}}{V|p+U|}\right)\Big\}, \end{aligned} \tag{4.7.36}$$

which has the same form as the relative displacement given by Bilby, Cottrell & Swinden (1963) for the corresponding static, anti-plane problem.

5. Further mixed boundary-value problems

5.1. *Formulation of problems with three mixed conditions*

(*a*) '*Punch*' *problems*

A more difficult problem than the one discussed in § 4.1 for the half-space $x_3 > 0$ is obtained by imposing the boundary conditions

$$\left.\begin{aligned} c_{i3kl}u_{k,l}(t,x) &= 0 \quad (x_3 = 0,\ x \notin S(t)),\\ u_i(t,x) &= w_i(t,x_\alpha) \quad (x \in S(t)),\end{aligned}\right\} \tag{5.1.1}$$

where $S(t)$ is the expanding ellipse (4.1.2) and $w_i(t,x_\alpha)$ are homogeneous functions of degree $n+1$. For the sake of giving the problem a concise name, it is termed here a 'punch' problem because it contains the dynamical generalization of the problem of static indentation by a perfectly rough punch for which the transverse displacement under the punch is prescribed, in fact, to be zero.

The problem would be solved by the results of § 2 if the traction vector $T(t,x_\alpha)$ were known over $S(t)$, and a possible approach is to set up an integral equation for $T(t,x_\alpha)$. Such an equation may be obtained from the representation (2.1.35), coupled with the argument given near the beginning of § 4.1. Thus, in analogy with equation (4.1.12), we have

$$\partial_t^{(n+2)}u(t,x) = -\frac{(2\pi)^{\frac{1}{2}}}{t}\oint_{|\eta|=1} \mathrm{d}s\,\mathrm{Re}\left\{\left(-\frac{\eta_\lambda x_\lambda}{t}\right)^{n+2} \times B\left(-\frac{\eta_\lambda x_\lambda}{t}+0\mathrm{i},\eta_\alpha\right)F^{(n+2)}\left(\frac{\eta_\lambda x_\lambda}{t}-0\mathrm{i},\eta\right)\right. \quad (x\in S(t)), \tag{5.1.2}$$

in which the sectionally holomorphic function $F^{(n+2)}(z,\eta)$ is defined by equation (2.1.36) and the matrix B is hermitian. The static limit of the problem is obtained by letting the density ρ of the half-space tend to zero, and in this limit the matrix B reduces to a function of η only. The static equation corresponding to (5.1.2) with $n = -2$ was derived directly by Willis (1971 *a*) and solutions were constructed, using properties of the Radon transform, for the case in which $u(x)$ was a polynomial, the half-space isotropic and the surface S a circle. A corresponding approach can be made to (5.1.2) directly if $w_i(t,x_\alpha)$ are (homogeneous) polynomials of degree $n+1$, for then the left side of (5.1.2) is zero and a solution would be obtained if a Radon transform $\check{T}(p,\eta)$ could be constructed, with a sufficient number of arbitrary constants, for which

$$\mathrm{Re}\{(-p)^{n+2}B(-p+0\mathrm{i},\eta_\alpha)\,F^{(n+2)}(p-0\mathrm{i},\eta)\} = 0 \quad (|p| < \omega_0(\eta)), \tag{5.1.3}$$

where $\omega_0(\eta)$ is defined by equation (4.4.4). This matrix Hilbert problem does not have a closed solution except in the static limit, but the construction of systematic approximations, and their use in solving the 'punch' problem, will be discussed in following sections.

The two-dimensional 'punch' problem is obtained by taking $S(t)$ to be the expanding strip (4.2.1) and $w_i(t,x_\alpha)$ to depend only upon $(t,\nu_\alpha x_\alpha)$; similar reasoning to that used in § 4.2 leads in this case, in analogy with equation (4.2.4), to the equation

$$\partial_t^{(n+2)}u(t,x) = \frac{2(2\pi)^{-\frac{1}{2}}}{t}\,\mathrm{Im}\left\{-\left(\frac{\nu_\lambda x_\lambda}{t}\right)^{n+2}B\left(-\frac{\nu_\lambda x_\lambda}{t}+0\mathrm{i},\nu_\alpha\right)G^{(n+1)}\left(\frac{\nu_\lambda x_\lambda}{t}-0\mathrm{i},\nu\right)\right\} \quad (x\in S(t)), \tag{5.1.4}$$

where $G^{(n+1)}(z, \nu)$ is defined by equation (2.3.8). Thus, upon setting

$$\partial_t^{(n+2)} u(t, x) = \frac{1}{t} w^{(n+2)}(p) \quad (x \in S(t)), \tag{5.1.5}$$

where $p = \nu_\lambda x_\lambda / t$, equation (5.1.4) implies the Hilbert problem

$$2(2\pi)^{-\frac{1}{2}} \operatorname{Im}\{(-p)^{n+2} B(-p+0\mathrm{i}, \nu_\alpha) G^{(n+1)}(p-0\mathrm{i}, \nu)\} = w^{(n+2)}(p) \quad (|p| < V), \tag{5.1.6}$$

whose solution is closely related to that of (5.1.3).

(*b*) *Interfacial cracks*

The problem strictly dual to the 'punch' problem described above would be that for a half-space $x_3 > 0$, with tractions prescribed over $S(t)$ and zero displacement prescribed over its complement. However, a generalization of this problem, which is of the same type and has more obvious practical value, concerns a crack occupying the part $S(t)$ of the interface $x_3 = 0$ between two dissimilar half-spaces, $x_3 > 0$ and $x_3 < 0$, which are perfectly bonded together over the remainder of their interface. This generalization will be discussed below, the strict dual of the 'punch' problem being obtainable by taking the half-space $x_3 < 0$ to be rigid.

The formulation of the interfacial crack problem follows precisely that given in § 4.5 for a crack in a homogeneous medium, at least as far as equation (4.5.4). The matrix which appears in (4.5.4) is now hermitian but not symmetric, so that the solution of (4.5.4) in the present context will differ in form from the one given in § 4.5, however. In the static limit, the matrix A^{-1} reduces to a function of η only and this limiting case has been studied directly by Willis (1972). A similar approach to that suggested for the 'punch' problem may be followed, if the traction vector $T(t, x_\alpha)$ is prescribed to be a (homogeneous) polynomial of degree n over $S(t)$, for then the left side of (4.5.4) is zero and the problem would be solved if a Radon transform could be constructed, with an appropriate number of arbitrary constants, for which

$$\operatorname{Re}\{(-p)^{n+1} A^{-1}(-p+0\mathrm{i}, \eta_\alpha) F^{(n+3)}(p-0\mathrm{i}, \eta)\} = 0 \quad (|p| < \omega_0(\eta)). \tag{5.1.7}$$

This Hilbert problem has the same structure as (5.1.3); its solution, and the solution of the crack problem, will be discussed in following sections.

The two-dimensional interfacial crack problem may be formulated similarly; we take $S(t)$ to be the expanding strip (4.4.10) and

$$T(t, x_\alpha) = t^n T(p), \tag{2.3.2}$$

where $p = \nu_\lambda x_\lambda / t$ and $T(p)$ is given for $|p| < V$. In this case, upon setting

$$\partial_t^{(n+1)} T(t, x_\alpha) = T^{(n+1)}(p)/t \tag{4.5.10}$$

equation (3.2.7) implies

$$2(2\pi)^{-\frac{3}{2}}(-1)^n \operatorname{Im}\{p^{n+1} A^{-1}(-p+0\mathrm{i}, \nu_\alpha) G^{(n+2)}(p-0\mathrm{i}, \nu)\} = T^{(n+1)}(p) \quad (|p| < V), \tag{5.1.8}$$

where $G^{(n+2)}(z, \nu)$ is defined by equation (3.2.4). This Hilbert problem has the same form as (5.1.6) and its solution is closely related to that of (5.1.7). The static limit of (5.1.8), for which A^{-1} reduces to a constant, has been obtained and studied directly by Willis (1971 *b*).

5.2. *Solution of the Hilbert problem for isotropic half-spaces*

Attention will now be directed towards solving the Hilbert problem (5.1.3) which may also be expressed in the form

$$B^T(-p+0\mathrm{i}, \eta_\alpha) F_+^{(n+2)}(p, \eta) - B(-p+0\mathrm{i}, \eta_\alpha) F_-^{(n+2)}(p, \eta) = 0 \quad (|p| < \omega_0(\eta)), \tag{5.2.1}$$

in view of the hermitian property of the matrix B and the fact that the Radon transform $\check{T}(p, \eta_\alpha)$ from which $F(z, \eta)$ is derived must be real. From its definition (2.1.36) it is clear that a solution of (5.2.1) must be sought for which

$$F^{(n+2)}(z,\eta) = O(z^{-n-3}) \quad (|z| \to \infty) \tag{5.2.2}$$

and

$$F^{(n+2)}(z,\eta) = O(z \mp \omega_0(\eta))^{-n-2} \quad (z \to \pm \omega_0(\eta)). \tag{5.2.3}$$

The problem will be approached by first attempting to construct a fundamental matrix $\phi(z, \eta)$ (Vekua, 1966), for which

$$B^T(-p+0\mathrm{i}, \eta_\alpha)\, \phi_+(p,\eta) - B(-p+0\mathrm{i}, \eta_\alpha)\, \phi_-(p,\eta) = 0 \quad (|p| < \omega_0(\eta)). \tag{5.2.4}$$

and

$$\phi(z,\eta) = O(1) \quad (|z| \to \infty \quad \text{or} \quad z \to \pm\omega_0(\eta)). \tag{5.2.5}$$

The matrix $\phi(z, \eta)$ is also required to be non-singular. In fact, by equating the determinants of the two terms in (5.2.4) it follows that $\det \phi(z, \eta)$ is entire and therefore must be constant if (5.2.5) is satisfied. If $\phi(z, \eta)$ were known, an appropriate form for $F^{(n+2)}(z, \eta)$ would be

$$F^{(n+2)}(z,\eta) = \phi(z,\eta)\, P_{n+1}(z,\eta)\, [z^2 - (\omega_0(\eta))^2]^{-n-2}, \tag{5.2.6}$$

where $P_{n+1}(z, \eta)$ is a column vector where components are polynomials of degree $n+1$ in z.

The two-dimensional problem is also soluble in terms of $\phi(z, \nu)$, with $\omega_0(\nu) = V$, for, upon setting

$$(-z)^{n+2}\, G^{(n+1)}(z,\nu) = \phi(z,\nu)\, H(z,\nu)\, (z^2 - V^2)^{-\frac{1}{2}}, \tag{5.2.7}$$

equation (5.1.6) implies

$$B^T(-p+0\mathrm{i}, \nu_\alpha)\, \phi_+(p,\nu)\, [H_+(p,\nu) - H_-(p,\nu)] = -(2\pi)^{\frac{1}{2}}\, (V^2 - p^2)^{\frac{1}{2}}\, w^{(n+2)}(p) \quad (|p| < V). \tag{5.2.8}$$

The most general admissible solution of (5.2.8) is

$$H(z,\nu) = (2\pi)^{-\frac{1}{2}}\, \mathrm{i} \int_{-V}^{V} \frac{(V^2-p^2)^{\frac{1}{2}}}{p-z}\, \phi_+^{-1}(p,\nu)\, B^{-T}(-p+0\mathrm{i}, \nu_\alpha)\, w^{(n+2)}(p)\, \mathrm{d}p + P_{2n+3}(z)\, (z^2 - V^2)^{-n-1}, \tag{5.2.9}$$

where P_{2n+3} is a polynomial of degree $2n+3$ in z, for then $G^{(n+1)}(z, \nu)$ is $O(z^{-n-2})$ as $|z| \to \infty$ and is $O(z \mp V)^{-n-\frac{3}{2}}$ as $z \to \pm V$. The coefficients in $P_{2n+3}(z)$ are not all arbitrary, as they must be chosen so that the right side of (5.2.9) is $O(z^{n+2})$ as $z \to 0$. In particular, if $w(t, \nu_\alpha x_\alpha)$ is a polynomial of degree $n+1$, so that $w^{(n+2)}(p) = 0$, then $P_{2n+3}(z) = (-z)^{n+2} P_{n+1}(z)$ and

$$G^{(n+1)}(z,\nu) = \phi(z,\nu)\, P_{n+1}(z)\, (z^2 - V^2)^{-n-\frac{3}{2}}. \tag{5.2.10}$$

The arbitrary coefficients in $P_{n+1}(z)$ suffice to generate any homogeneous polynomial $w(t, \nu_\alpha x_\alpha)$ of degree $n+1$ and the solution of the two-dimensional problem may be completed as indicated in § 4.2 for the frictionless 'punch' problem.

The complete solution of the three-dimensional problem is more complicated, owing to the need to select the polynomial vector $P_{n+1}(z, \eta)$ so that $F(z, \eta)$ yields a function $\check{T}(p, \eta_\alpha)$ which is indeed a Radon transform. This problem has not yet been resolved in generality even in the static limit (see Willis 1971 *a*); consequently, in the sequel we specialize to an isotropic half-space and to the circular region $S(t)$:

$$x_3 = 0, \quad x_1^2 + x_2^2 \leqslant V^2 t^2, \tag{5.2.11}$$

for which a systematic construction can be developed. In the remainder of this section, therefore, the matrix $B(-p+0\mathrm{i}, \eta_\alpha)$ will be taken to have the special form (2.6.13) and systematic approximations to $\phi(z,\eta)$ will be generated for this case.

When the matrix $B(-p+0\mathrm{i}, \eta_\alpha)$ has the form (2.6.13), the Hilbert problem uncouples partly, and a matrix $\phi(z,\eta)$ can be sought for which

$$\phi(z,\eta) = \begin{bmatrix} -\eta_2 & \mathrm{i}\eta_1 g(z) & -\mathrm{i}\eta_1 g(-z) \\ \eta_1 & \mathrm{i}\eta_2 g(z) & -\mathrm{i}\eta_2 g(-z) \\ 0 & h(z) & h(-z) \end{bmatrix}, \tag{5.2.12}$$

where the sectionally holomorphic functions $g(z)$, $h(z)$ are to be determined. The form of the third column of $\phi(z,\eta)$ is possible because the matrix $B(-p+0\mathrm{i}, \eta_\alpha)$ is an even function of p. To construct $g(z)$, $h(z)$, set

$$\psi(z) = \begin{bmatrix} \mathrm{i}g(z) & -\mathrm{i}g(-z) \\ h(z) & h(-z) \end{bmatrix}. \tag{5.2.13}$$

Then $\psi(z)$ satisfies the equation

$$B_1^T \psi_+ - B_1 \psi_- = 0, \quad |p| < V, \tag{5.2.14}$$

where

$$B_1(p) = \begin{bmatrix} b(-p+0\mathrm{i}) & -\mathrm{i}d(-p+0\mathrm{i}) \\ \mathrm{i}d(-p+0\mathrm{i}) & e(-p+0\mathrm{i}) \end{bmatrix}. \tag{5.2.15}$$

Guided by the construction employed by Willis (1971 *a*) in the static limit, an approximate $\psi(z)$ will be found by first noting that the eigenvalue problem

$$(B_1^T - \lambda B_1)\, U = 0 \tag{5.2.16}$$

has solutions

$$U_1 = \begin{bmatrix} \mathrm{i} \\ (b/e)^{\frac{1}{2}} \end{bmatrix}, \quad \lambda_1 = \frac{(be)^{\frac{1}{2}} + d}{(be)^{\frac{1}{2}} - d}, \tag{5.2.17}$$

and

$$U_2 = \begin{bmatrix} -\mathrm{i} \\ (b/e)^{\frac{1}{2}} \end{bmatrix}, \quad \lambda_2 = \frac{(be)^{\frac{1}{2}} - d}{(be)^{\frac{1}{2}} + d}. \tag{5.2.18}$$

Now if $(b/e)^{\frac{1}{2}}$ were a polynomial in p, a particular solution of (5.2.14) would be

$$\psi(z) = [U_1 f(z),\ U_2 f(-z)], \tag{5.2.19}$$

so long as

$$\lambda_1(p) f_+(p) - f_-(p) = 0 \quad (|p| < V), \tag{5.2.20}$$

that is

$$f(z) = \exp\left\{-\frac{1}{2\pi\mathrm{i}} \int_{-V}^{V} \frac{\ln[\lambda_1(p)]\,\mathrm{d}p}{p-z}\right\}. \tag{5.2.21}$$

If $(b/e)^{\frac{1}{2}}$ is not a polynomial, an approximate solution of (5.2.14) is obtained by approximating $(b/e)^{\frac{1}{2}}$ by a polynomial in p on the interval $[-V, V]$. If, therefore

$$(b/e)^{\frac{1}{2}} \simeq K \prod_{i=1}^{n} (1 - p^2/c_i^2) \tag{5.2.22}$$

on $[-V, V]$, an approximate solution of (5.2.14) is

$$\psi(z) = \begin{bmatrix} 1 & 0 \\ 0 & \pi(1 - z^2/c_i^2) \end{bmatrix} \psi_0(z), \tag{5.2.23}$$

where

$$\psi_0(z) = \begin{bmatrix} \mathrm{i}g_0(z) & -\mathrm{i}g_0(-z) \\ h_0(z) & h_0(-z) \end{bmatrix}, \tag{5.2.24}$$

and
$$g_0(z) = f(z), \quad h_0(z) = Kf(z). \tag{5.2.25}$$

The approximation (5.2.22) involves only p^2 as $(b/e)^{\frac{1}{2}}$ is an even function of p.

The approximate solution (5.2.23) does not yield a fundamental matrix because it is $O(z^{2n})$ as z tends to infinity and also its determinant vanishes at $z = \pm c_i$, $i = 1, 2, \ldots, n$. However, a fundamental matrix can be constructed from it as follows. First, define

$$\psi_0^*(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 - z^2/c_1^2 \end{bmatrix} \psi_0(z). \tag{5.2.26}$$

The determinant of the matrix $\psi_0^*(z)$ contains $1 - z^2/c_1^2$ as a factor and the object is to eliminate this by operations on the columns of $\psi_0^*(z)$, corresponding to post-multiplication by another matrix. This is effected by setting

$$\psi_1(z) = \psi_0^*(z) \begin{bmatrix} g_0(-c_1)/(1 - z/c_1) & g_0(c_1)/(1 + z/c_1) \\ g_0(c_1)/(1 - z/c_1) & g_0(-c_1)/(1 + z/c_1) \end{bmatrix} \begin{bmatrix} 1 - k_1 z & k_1 z \\ -k_1 z & 1 + k_1 z \end{bmatrix}. \tag{5.2.27}$$

Then $\det \psi_1(z)$ is constant and

$$\psi_1(z) = \begin{bmatrix} \mathrm{i}g_1(z) & -\mathrm{i}g_1(-z) \\ h_1(z) & h_1(-z) \end{bmatrix}, \tag{5.2.28}$$

where
$$\left.\begin{aligned} g_1(z) &= (1 - k_1 z)\, g_1^*(z) + k_1 z g_1^*(-z), \\ h_1(z) &= (1 - k_1 z)\, h_1^*(z) - k_1 z h_1^*(-z), \end{aligned}\right\} \tag{5.2.29}$$

and
$$\left.\begin{aligned} g_1^*(z) &= \frac{g_0(z)\, g_0(-c_1) - g_0(-z)\, g_0(c_1)}{1 - z/c_1}, \\ h_1^*(z) &= (1 + z/c_1)\, [h_0(z)\, g_0(-c_1) + h_0(-z)\, g_0(c_1)]. \end{aligned}\right\} \tag{5.2.30}$$

Further, if
$$h_0(z) \sim h_0(\infty) + \delta/z \quad (|z| \to \infty), \tag{5.2.31}$$

each component of $\psi_1(z)$ is bounded as z tends to infinity if k_1 is chosen so that

$$k_1 = \frac{[g_0(c_1) + g_0(-c_1)]\, h_0(\infty)}{2c_1\{[g_0(c_1) + g_0(-c_1)]\, h_0(\infty) - \delta[g_0(c_1) - g_0(-c_1)]/c_1\}}. \tag{5.2.32}$$

We may now define
$$\psi_1^*(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 - z^2/c_2^2 \end{bmatrix} \psi_1(z) \tag{5.2.33}$$

and repeat the reduction. After n steps, a matrix $\psi_n(z)$ is obtained, which is equal to $\psi_0(z)$ post-multiplied by a matrix whose components are meromorphic, is $O(1)$ at infinity and has constant determinant. The matrix $\psi_n(z)$ thus satisfies the Hilbert problem exactly when $(b/e)^{\frac{1}{2}}$ is the polynomial on the right side of (5.2.22) and may therefore be taken as an approximate fundamental matrix of (5.2.16), from which an approximate $\phi(z, \eta)$ follows by substitution into (5.2.12).

5.3. *Solution of two-dimensional 'punch' and crack problems*

The basic solution of the two-dimensional 'punch' problem has already been outlined and is embodied in equation (5.2.9). Equations (5.2.7) and (5.2.9) give

$$(-z)^{n+2}\, G^{(n+1)}(z, \nu) = \phi(z, \nu) \bigg\{ (2\pi)^{-\frac{1}{2}}\, \mathrm{i}\, (z^2 - V^2)^{-\frac{1}{2}} \int_{-V}^{V} \frac{(V^2 - p^2)^{\frac{1}{2}}}{p - z}\, \phi_+^{-1}(p, \nu)\, B^{-T}(-p + 0\mathrm{i}, \nu_\alpha)$$
$$\times\, w^{(n+2)}(p)\, \mathrm{d}p + P_{2n+3}(z, \nu)\, (z^2 - V^2)^{-n-\frac{3}{2}} \bigg\}. \tag{5.3.1}$$

The solution is completed by first restricting the coefficients in P_{2n+3} so that $G^{(n+1)}(z,\nu)$ is not singular at $z=0$ and then integrating $n+1$ times with respect to z remembering that $G(z,\nu)\to 0$ as $z\to\infty$. The traction vector $T(t,x_\alpha)=t^n T(p)$ then follows by use of the Plemelj formula

$$T(p) = G_+(p,\nu) - G_-(p,\nu). \tag{5.3.2}$$

The remaining $n+1$ arbitrary constants are fixed by reverting to the representation (2.3.10) with $x_3=0$, integrating $n+1$ times with respect to t with $G^{(n+1)}(z,\nu)$ given by (5.3.1) and then adjusting the constants so that the desired $u(t,x_\alpha)$ is obtained over $S(t)$. Details will not be pursued here, but an example will be given later.

The solution of the two-dimensional interfacial crack problem is similar. Let $\chi(z,\nu)$ denote a fundamental matrix of the Hilbert problem (5.1.7) and set

$$z^{n+1}G^{(n+2)}(z,\nu) = \chi(z,\nu)\,H(z,\nu)\,(z^2-V^2)^{-\frac{1}{2}}. \tag{5.3.3}$$

Substitution of (5.3.3) into (5.1.8) then gives

$$T^{(n+1)}(p) = (2\pi)^{-\frac{7}{2}}(-1)^n(V^2-p^2)^{-\frac{1}{2}}A^{-T}(-p+0\mathrm{i},\nu_\alpha)\,\chi_+(p,\nu) \\ \times[H_+(p,\nu)-H_-(p,\nu)] \quad (|p|<V), \tag{5.3.4}$$

which must be solved subject to the conditions

$$\left.\begin{aligned} H(z,\nu) &= O(z^{-1}) \quad (|z|\to\infty),\\ &= O(z\mp V)^{-n-\frac{3}{2}+\delta} \quad (z\to\pm V),\end{aligned}\right\} \tag{5.3.5}$$

where $0<\delta<1$, since the relative displacement of the crack faces is zero at the crack edges, $p=\pm V$. Hence,

$$H(z,\nu) = (2\pi)^{\frac{5}{2}}\mathrm{i}(-1)^{n+1}\int_{-V}^{V}\frac{(V^2-p^2)^{\frac{1}{2}}}{p-z}\chi_+^{-1}(p,\nu)\,A^T(-p+0\mathrm{i},\nu_\alpha)\,T^{(n+1)}(p)\,\mathrm{d}p \\ +P_{2n+1}(z)\,(z^2-V^2)^{-n-1}, \tag{5.3.6}$$

where $P_{2n+1}(z)$ is a polynomial of degree $2n+1$ in z. The general solution for $G^{(n+2)}(z,\nu)$ now follows from (5.3.6) and (5.3.3), and it is clear that the coefficients in P_{2n+1} must be restricted so that $H(z,\nu)=O(z^{n+1})$ as $z\to 0$. In particular, if $T(t,x_\alpha)$ is a (homogeneous) polynomial of degree n in $(t,\nu_\alpha x_\alpha)$, then $T^{(n+1)}(p)=0$ and

$$G^{(n+2)}(z,\nu) = \chi(z,\nu)\,P_n(z)\,(z^2-V^2)^{-n-\frac{3}{2}}, \tag{5.3.7}$$

where $P_n(z)$ is a polynomial vector of degree n, whose coefficients must be adjusted so that the correct $T(t,x_\alpha)$ is obtained.

5.4. *Solution of three-dimensional 'punch' and crack problems*

As remarked in § 5.2, completion of the solution of three-dimensional 'punch' and crack problems is more difficult than for the corresponding two-dimensional problems, owing to the need to ensure that functions $\check{T}(p,\eta_\alpha)$ or $\check{b}(p,\eta_\alpha)$ are constructed which are indeed Radon transforms. It has not so far proved possible to resolve this problem even in the static case (Willis 1971*a*, 1972) without exploiting in some detail the structure of the matrices $B(-p+0\mathrm{i},\eta_\alpha)$, $A^{-1}(-p+0\mathrm{i},\eta_\alpha)$, and solutions have been obtained only for problems involving isotropic (or transversely isotropic) half-spaces.

Dealing first with the 'punch' problem, for an isotropic half-space the fundamental matrix takes the form (5.2.12). If the case is now considered for which $u(t, x_\alpha)$ is a homogeneous polynomial of degree $n+1$ over $S(t)$, the solution for $F^{(n+2)}(z, \eta)$ has the form (5.2.6) and the problem is to select the polynomial vector $P_{n+1}(z, \eta)$ so that the Plemelj formula

$$T(p, \eta_\alpha) = F_+(p, \eta) - F_-(p, \eta) \tag{5.4.1}$$

yields a real vector $\check{T}(p, \eta_\alpha)$ which satisfies the necessary and sufficient conditions (A 2), (A 3) so that it is a Radon transform. To this end, it is convenient to define a different fundamental matrix by replacing the second and third columns of $\phi(z, \eta)$ by their sum and difference respectively, to yield a new matrix

$$\phi(z, \eta) = [U_0(\eta), V_1(z, \eta), V_2(z, \eta)], \tag{5.4.2}$$

where

$$U_0(\eta) = \begin{bmatrix} -\eta_2 \\ \eta_1 \\ 0 \end{bmatrix}, \quad V_1(z, \eta) = \begin{bmatrix} \mathrm{i}\eta_1[g(z) - g(-z)] \\ \mathrm{i}\eta_2[g(z) - g(-z)] \\ h(z) + h(-z) \end{bmatrix}, \quad V_2(z, \eta) = \begin{bmatrix} \mathrm{i}\eta_1[g(z) + g(-z)] \\ \mathrm{i}\eta_2[g(z) + g(-z)] \\ 0 \end{bmatrix}. \tag{5.4.3}$$

Now $\bar{\phi}(z, \eta)$ is also a fundamental matrix and so any column of $\bar{\phi}(z, \eta)$ must be expressible as a linear combination of those of $\phi(z, \eta)$. Consideration of the even and odd properties of each component now shows that $\bar{V}_1(z, \eta)$ must be proportional to $V_1(z, \eta)$ and $\bar{V}_2(z, \eta)$ must be proportional to $V_2(z, \eta)$. Hence, by scaling $V_1(z, \eta)$, $V_2(z, \eta)$, a fundamental matrix $\phi(z, \eta)$ can be selected so that it has the form (5.4.2), and also $\bar{\phi}(z, \eta) = \phi(z, \eta)$.

With $\phi(z, \eta)$ given by (5.4.2) and $F^{(n+3)}(z, \eta)$ as in (5.2.6) the method of selection of the polynomial vector $P_{n+1}(z, \eta)$ parallels that which was used by Willis (1971 *a*) in the static case, but is a little less explicit and also rather more compact. Consider the expression

$$F^{(n+2)}(z, \eta) = (\eta_1 + \mathrm{i}\eta_2)^m \{\mathrm{i}\, U_0(\eta)\, R_{n+1}(z) + V_1(z, \eta)\, S_{n+1}(z) + V_2(z, \eta)\, T_{n+1}(z)\} (z^2 - V^2)^{-n-2}, \tag{5.4.4}$$

where R_{n+1}, S_{n+1} and T_{n+1} are scalar polynomials of degree $n+1$. The definition (2.1.36) of $F^{(n+2)}(z, \eta)$ and the even property (A 2) of the Radon transform implies that $F^{(n+2)}(z, \eta)$ is either an even or an odd function of (z, η) and this immediately places restrictions on the polynomials. Next, the condition (A 3) may be expressed, upon the use of the Plemelj formula (5.4.1), in the form

$$\oint z^k F(z, \eta)\, \mathrm{d}z = P_k(\eta), \tag{5.4.5}$$

where $P_k(\eta)$ stands for a polynomial of degree $\leqslant k$ in η, and the integral is taken around any contour enclosing the segment $[-V, V]$ of the real axis. This may be reduced, by integration by parts $n+2$ times, to the form

$$\oint z^{n+2+k} F^{(n+2)}(z, \eta)\, \mathrm{d}z = P_k(\eta), \tag{5.4.6}$$

which places restrictions directly upon the polynomials R_{n+1}, S_{n+1} and T_{n+1}. The application of (5.4.6) is simplified by observing that, if $m \geqslant 1$, the term in brackets in (5.4.4) may be combined with a factor $(\eta_1 + \mathrm{i}\eta_2)$ to give

$$(-\eta_2\, U_0 R_{n+1} + \eta_1 V_1 S_{n+1} + \eta_1 V_2 T_{n+1}) + \mathrm{i}(\eta_1\, U_0 R_{n+1} + \eta_2 V_1 S_{n+1} + \eta_2 V_2 T_{n+1}), \tag{5.4.7}$$

which may be compared with corresponding terms in equation (6.10) of Willis (1971*a*). These groups of terms are convenient in that the first bracket may be re-written

$$\begin{bmatrix} R_{n+1}-\eta_1^2(R_{n+1}-\mathrm{i}g^{\mathrm{o}}S_{n+1}-\mathrm{i}g^{\mathrm{e}}T_{n+1}) \\ -\eta_1\eta_2(R_{n+1}-\mathrm{i}g^{\mathrm{o}}S_{n+1}-\mathrm{i}g^{\mathrm{e}}T_{n+1}) \\ \eta_1(h^{\mathrm{e}}S_{n+1}+h^{\mathrm{o}}T_{n+1}) \end{bmatrix}, \tag{5.4.8}$$

where g^{o} stands for the odd function $g(z)-g(-z)$, g^{e} stands for the even function $g(z)+g(-z)$, and so on; the second bracket may be re-written similarly.

When $m = 0$, (5.4.6) is satisfied for all polynomials R_{n+1}, S_{n+1}, T_{n+1} which conform with the even property (A 2). For $m \geqslant 1$, however, (5.4.6) implies

$$\oint z^{n+2+k}R_{n+1}(z)\frac{\mathrm{d}z}{(z^2-V^2)^{n+2}} = 0 \quad (k = 0, 1, \ldots, m-2 \quad (m \geqslant 2)), \tag{5.4.9}$$

$$\oint z^{n+2+k}\{R_{n+1}(z)-\mathrm{i}g^{\mathrm{o}}(z)\,S_{n+1}(z)-\mathrm{i}g^{\mathrm{e}}(z)\}\frac{\mathrm{d}z}{(z^2-V^2)^{n+2}} = 0 \quad (k = 0, 1, \ldots, m), \tag{5.4.10}$$

$$\oint z^{n+2+k}\{h^{\mathrm{e}}(z)\,S_{n+1}(z)+h^{\mathrm{o}}(z)\,T_{n+1}(z)\}\frac{\mathrm{d}z}{(z^2-V^2)^{n+2}} = 0 \quad (k = 0, 1, \ldots, m-1). \tag{5.4.11}$$

Now deforming the contour of integration in (5.4.9) to infinity shows that $R_{n+1}(z)$ must actually have degree at most $n-m+2$. A similar operation on (5.4.11) coupled with the observation that $h^{\mathrm{e}}(z) = O(1)$ as $z\to\infty$, while $h^{\mathrm{o}}(z)\to 0$ as $z\to\infty$ since it is an odd function, shows that $S_{n+1}(z)$ has degree at most $n-m+1$ and $T_{n+1}(z)$ has degree at most $n-m+2$. Therefore, only values of m in the range $0 \leqslant m \leqslant n+2$ may be admitted in (5.4.4). With the above restrictions, together with the appropriate limitations to odd or even functions, equation (5.4.10) is satisfied automatically for all k except either $k = m-1$ or $k = m$. Some careful accounting, allowing for both the real and imaginary parts of the Radon transforms generated for $m \geqslant 1$, now shows that (5.4.4) yields a Radon transform with $\frac{3}{2}(n+2)(n+3)$ constants unspecified, which suffice exactly to match a homogeneous vector polynomial of degree $n+1$ in (t, x_α). The solution to the 'punch' problem is thus obtained in principle, the only task remaining being the evaluation of a number of definite integrals.

The solution of the crack problem is obtained similarly. The Hilbert problem (5.1.7) with $\omega_0(\eta) = V$, may be expressed in the form

$$A^{-T}(-p+0\mathrm{i},\eta_\alpha)\,F_+^{(n+3)}(p,\eta)-A^{-1}(-p+0\mathrm{i},\eta_\alpha)\,F_-^{(n+3)}(p,\eta) = 0 \quad (|p| < V), \tag{5.4.12}$$

since $F^{(n+3)}(z,\eta)$ is defined by equation (3.1.10) and $\check{b}(p,\eta_\alpha)$ is real. As $\check{b}(p,\eta_\alpha)$ tends to zero as p tends to $\pm V$, the Hilbert problem must be solved subject to the conditions

$$\left.\begin{aligned} F^{(n+3)}(z,\eta) &= O(z\mp V)^{-n-2}, \quad (z\to\pm V), \\ &= O(z^{-n-4}) \quad (z\to\infty). \end{aligned}\right\}$$

Therefore, if $\chi(z,\eta)$ is a fundamental matrix of (5.4.12) the most general admissible expression for $F^{(n+3)}(z,\eta)$ is

$$F^{(n+3)}(z,\eta) = \chi(z,\eta)\,P_n(z,\eta)\,(z^2-V^2)^{-n-2}, \tag{5.4.13}$$

where $P_n(z,\eta)$ is a polynomial of degree n in z, which must be restricted so that the Plemelj formula

$$\check{b}(p,\eta_\alpha) = F_+(p,\eta)-F_-(p,\eta) \tag{5.4.14}$$

yields a real Radon transform $\check{b}(p,\eta_\alpha)$.

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For isotropic half-spaces the matrix $A^{-1}(-p+0\mathrm{i}, \eta_\alpha)$ has exactly the same form as the matrix $B(-p+0\mathrm{i}, \eta_\alpha)$ given by equation (2.6.13) and $\chi(z, \eta)$ can be taken to have the same structure as (5.4.4) but with the vectors $V_1(z, \eta)$, $V_2(z, \eta)$ being obtained from $A^{-1}(-p+0\mathrm{i}, \eta_\alpha)$ instead of $B(-p+0\mathrm{i}, \eta_\alpha)$. With $\chi(z, \eta)$ defined in this way, the polynomial vector $P_n(z, \eta)$ may be chosen as in the 'punch' problem, so that

$$F^{(n+3)}(z, \eta) = (\eta_1+\mathrm{i}\eta_2)^m \{\mathrm{i}U_0(\eta)\, R_n(z) + V_1(z, \eta)\, S_n(z) + V_2(z, \eta)\, T_n(z)\}\,(z^2-V^2)^{-n-2}. \quad (5.4.15)$$

In analogy with the 'punch' problem, the restrictions that are placed on the polynomials R_n, S_n, T_n are that $F(z, \eta)$ must be an odd function of (z, η) and that for $m \geqslant 1$,

$$\oint z^{n+3+k} R_n(z) \frac{\mathrm{d}z}{(z^2-V^2)^{n+2}} = 0 \quad (k = 0, 1, \ldots, m-2 \quad (m \geqslant 2)), \quad (5.4.16)$$

$$\oint z^{n+3+k} \{R_n(z) - \mathrm{i}g^{\mathrm{o}}(z)\, S_n(z) - \mathrm{i}g^{\mathrm{e}}(z)\, T_n(z)\} \frac{\mathrm{d}z}{(z^2-V^2)^{n+2}} = 0 \quad (k = 0, 1, \ldots, m), \quad (5.4.17)$$

$$\oint z^{n+3+k} \{h^{\mathrm{e}}(z)\, S_n(z) + h^{\mathrm{o}}(z)\, T_n(z)\} \frac{\mathrm{d}z}{(z^2-V^2)^{n+2}} = 0 \quad (k = 0, 1, \ldots, m-1). \quad (5.4.18)$$

These conditions restrict $R_n(z)$ and $T_n(z)$ to have degrees at most $n-m+1$, and $S_n(z)$ to have degree at most $n-m$. Condition (5.4.17) also imposes one further condition, either when $k = m-1$ or when $k = m$, and (5.4.15) then contains just sufficient undetermined constants (with $0 \leqslant m \leqslant n+1$) to match an arbitrary homogeneous vector polynomial $T(t, x_\alpha)$ of degree n.

5.5. *Examples*

(*a*) *The two-dimensional 'punch' problem* $(n = -1)$

A simple example of a 'punch' problem is obtained by taking $S(t)$ to be the strip

$$x_3 = 0, \quad |x_1| < Vt, \quad (5.5.1)$$

so that $\nu_1 = 1$, $\nu_2 = 0$, and applying the mixed boundary conditions (5.1.1) with

$$\left.\begin{aligned} w_1(t, x_\alpha) &= w_1^0, \\ w_2(t, x_\alpha) &= 0, \\ w_3(t, x_\alpha) &= w_3^0, \end{aligned}\right\} \quad (5.5.2)$$

which define a problem of plane strain in the (x_1, x_3) plane. For this example, $n = -1$ and the solution (5.2.10) reduces to

$$G(z) = \mathrm{i}\phi(z)\, P_0 (z^2-V^2)^{-\frac{1}{2}}, \quad (5.5.3)$$

where P_0 is a constant vector. Equation (5.3.2) gives

$$T(p) = [\phi(p+0\mathrm{i}) + \phi(p-0\mathrm{i})]\, P_0 (V^2-p^2)^{-\frac{1}{2}} \quad (|p| < V), \quad (5.5.4)$$

and this real if P_0 is real, if $\phi(z)$ is chosen, as in § 5.4, so that $\overline{\phi}(z) = \phi(z)$. The constant vector P_0 is now fixed by substituting (5.5.3) into (2.3.10) to given an expression for $u(t, x)$ on the surface $x_3 = 0$ and equating this to w^0 over $S(t)$. Of course, $u(t, x)$ would be exactly constant over $S(t)$ if $\phi(z)$ were known exactly and substituting (5.5.3) with an approximate $\phi(z)$ into (2.3.10) will yield a displacement $u(t, x)$ whose variation over $S(t)$ will provide a check on the adequacy of the approximation to $\phi(z)$.

In evaluating the integral in (2.3.10), it is convenient to split it into two parts, so that

$$u(t, x)|_{x_3=0} = 2(2\pi)^{-\frac{1}{2}}\, \mathrm{Im}\left\{\int_{-x_1/t+0\mathrm{i}}^{0\mathrm{i}} \mathrm{d}\Omega\, B(\Omega, \nu)\, G(-\Omega)\right\} + u(t, 0), \quad (5.5.5)$$

and to choose the contour of integration for $u(t, 0)$ to be the positive imaginary axis. This gives

$$u(t,0) = 2(2\pi)^{-\frac{1}{2}}\,\mathrm{Re}\left\{\int_0^\infty \mathrm{d}p\, B(\mathrm{i}p)\, G(-\mathrm{i}p)\right\}. \tag{5.5.6}$$

Now since $\nu_1 = 1$, $\nu_2 = 0$, $u_2(t,x) = 0$ if $G_2(z) = 0$, and in this case the relevant part of $B(\Omega,\nu)$ is just $B_1(\Omega)$, which is defined by equation (5.2.15) as $B(\Omega,\nu)$ with the 2-components deleted. Correspondingly, the relevant part of $\phi(z)$ is

$$\psi(z) = \begin{bmatrix} \mathrm{i}g^{\mathrm{e}}(z) & \mathrm{i}g^{\mathrm{o}}(z) \\ h^{\mathrm{o}}(z) & h^{\mathrm{e}}(z) \end{bmatrix}, \tag{5.5.7}$$

which is obtained from the sum and difference of the columns of (5.2.13) and for which it may be assumed that $\overline{\psi}(z) = \psi(z)$. Equation (5.5.6) now gives

$$u(t,0) = 2(2\pi)^{-\frac{1}{2}}\,\mathrm{Re}\left\{\int_0^\infty \frac{\mathrm{d}p}{(V^2+p^2)^{\frac{1}{2}}} \begin{bmatrix} \mathrm{i}(bg^{\mathrm{e}}-dh^{\mathrm{o}}) & \mathrm{i}(bg^{\mathrm{o}}-dh^{\mathrm{e}}) \\ (eh^{\mathrm{o}}-dg^{\mathrm{e}}) & (eh^{\mathrm{e}}-dg^{\mathrm{o}}) \end{bmatrix} P_0\right\}, \tag{5.5.8}$$

where b, d and e are evaluated for $\Omega = \mathrm{i}p$ while g^{o}, g^{e}, h^{o}, h^{e} are evaluated for $z = -\mathrm{i}p$. Now b, d and e are given by equations (2.6.14) and are real when $\Omega = \mathrm{i}p$. Also it is easily deduced, using the property $\overline{\psi}(z) = \psi(z)$, that $g^{\mathrm{o}}(-\mathrm{i}p)$ and $h^{\mathrm{e}}(-\mathrm{i}p)$ are real, while $g^{\mathrm{e}}(-\mathrm{i}p)$ and $h^{\mathrm{o}}(-\mathrm{i}p)$ are imaginary. Hence, (5.5.8) reduces to

$$u(t,0) = 2(2\pi)^{-\frac{1}{2}}\int_0^\infty \frac{\mathrm{d}p}{(V^2-p^2)^{\frac{1}{2}}} \begin{bmatrix} \mathrm{i}(bg^{\mathrm{e}}-dh^{\mathrm{o}})\,P_1 \\ (eh^{\mathrm{e}}-dg^{\mathrm{o}})\,P_3 \end{bmatrix}, \tag{5.5.9}$$

where P_1 and P_3 are the 1- and 3-components of the vector P_0.

In order to compute $u(t,x)$, an approximate $\psi(z)$ must be constructed; for illustration, computations have been performed for the simplest approximation to the function $(b/e)^{\frac{1}{2}}$ which has the correct values when $\Omega = 0$ and when $\Omega = V$. Thus

$$(b/e)^{\frac{1}{2}} = \left(\frac{1-\Omega^2/\beta^2}{1-\Omega^2/\alpha^2}\right)^{\frac{1}{4}} \simeq 1 - \frac{\Omega^2}{c_1^2}, \tag{5.5.10}$$

where

$$c_1 = V\left\{1 - \left(\frac{1-V^2/\beta^2}{1-V^2/\alpha^2}\right)^{\frac{1}{4}}\right\}^{-\frac{1}{2}}. \tag{5.5.11}$$

This approximation requires just the one cycle of the reduction given explicitly in § 5.2. Since c_1 is real, $f(c_1)$ and $\bar{f}(c_1)$ are complex conjugates, k_1 is real and the $\psi(z)$ constructed satisfies automatically the condition $\overline{\psi}(z) = \psi(z)$. A further approximation was also made, to reduce the computing, by taking

$$(2\pi)^{-1}\ln[\lambda_1(p)] \equiv \kappa(p) \simeq \kappa_0 + \kappa_1 p^2, \tag{5.5.12}$$

with

$$\kappa_0 = (2\pi)^{-1}\ln[(\alpha^2+\beta^2)/(\alpha^2-\beta^2)], \quad \kappa_1 = [\kappa(V)-\kappa_0]/V^2. \tag{5.5.13}$$

This gave the closed expression

$$f(z) = \exp \mathrm{i}\left\{(\kappa_0+\kappa_1 z^2)\ln\left(\frac{z-V}{z+V}\right) + 2\kappa_1 Vz\right\} \tag{5.5.14}$$

in place of the integral (5.2.21). The variations in $u(t,x)$ along $S(t)$ are given in figures 2 and 3 when the resultant shear force F_1 is normalized to μ, and in figures 4 and 5 when the resultant normal force F_3 is normalized to μ. The normalization was effected by noting that the resultant force F is expressible as

$$F = \int_{-V}^{V} T(p)\,\mathrm{d}p = -\oint_C G(z)\,\mathrm{d}z, \tag{5.5.15}$$

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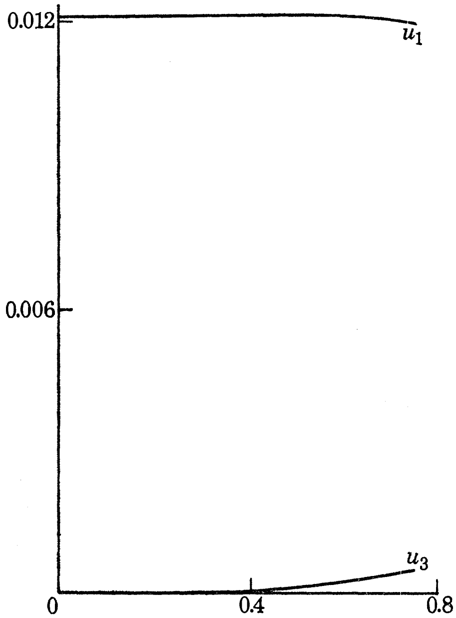

FIGURE 2. Plots of $u_1(t, x_1)$ and $u_3(t, x_1)$ for the 'flat punch' problem, with $F_1 = \mu$, $F_3 = 0$ and $V/\beta = 0.8$, using the approximation (5.5.10).

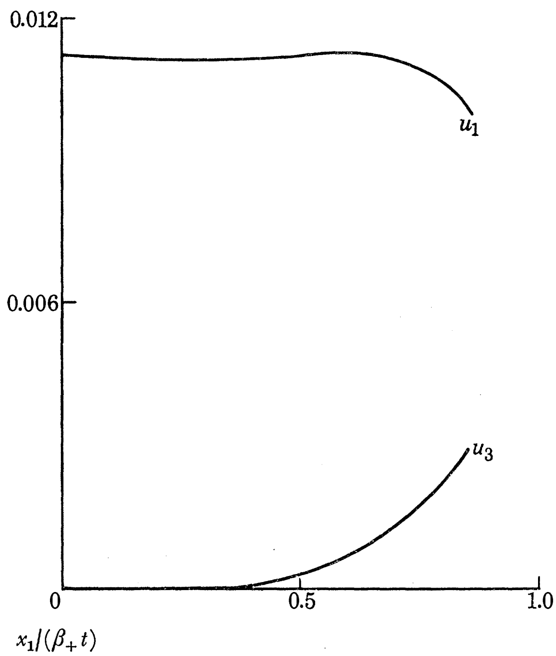

FIGURE 3. Plots of $u_1(t, x_1)$ and $u_3(t, x_1)$ for the 'flat punch' problem, with $F_1 = \mu$, $F_3 = 0$ and $V/\beta = 0.9$, using the approximation (5.5.10).

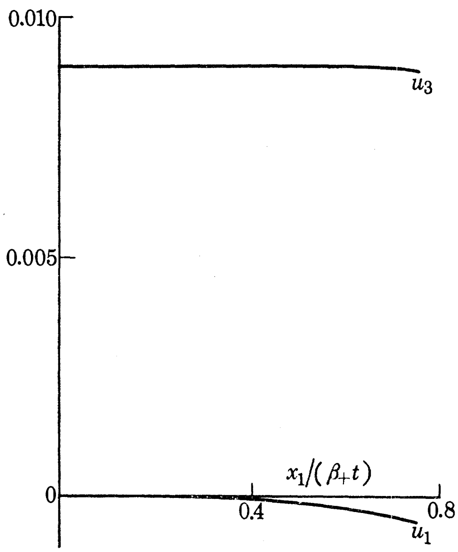

FIGURE 4. Plots of $u_1(t, x_1)$ and $u_3(t, x_1)$ for the 'flat punch' problem, with $F_1 = 0$, $F_3 = \mu$ and $V/\beta = 0.8$, using the approximation (5.5.10).

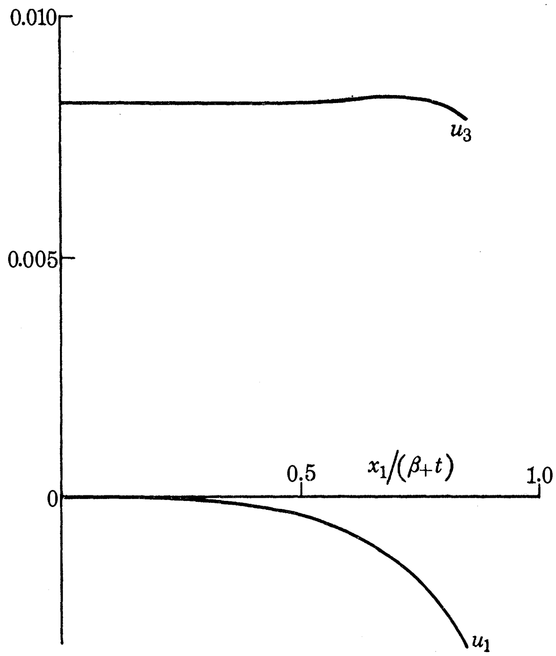

FIGURE 5. Plots of $u_1(t, x_1)$ and $u_3(t, x_1)$ for the 'flat punch' problem, with $F_1 = 0$, $F_3 = \mu$ and $V/\beta = 0.9$, using the approximation (5.5.10).

where C is any contour enclosing the segment $[-V, V]$ of the real axis, by use of the Plemelj formula (3.3.2). Now by deforming C to infinity, the two-dimensional reduction of (5.5.3), obtained by replacing $\phi(z)$ by $\psi(z)$, shows that

$$G(z) \sim \mathrm{i}\psi(\infty)\, P_0/z,$$

so that

$$F = 2\pi\psi(\infty)\, P_0. \tag{5.5.16}$$

It may be noted that $\psi(\infty)$ is a diagonal matrix since the functions $g^o(z)$, $h^o(z)$ are both odd and bounded, so that $P_1 = 0$ implies $F_1 = 0$ and $P_3 = 0$ implies $F_3 = 0$.

The approximation (5.5.10) is better for small values of V/β than for values approaching unity and figures 2 to 5 bear this out. It is interesting that the simple approximation (5.5.10) remains adequate for values of V/β as high as 0.8, as shown in figures 2 and 4, and hardly surprising that it is less good for $V/\beta = 0.9$, for which figures 3 and 5 indicate the need for further terms.

(b) *The circular crack problem* ($n = 0$)

For the circular crack problem, the matrix $A(\Omega, \eta_\alpha)$ is given as

$$A(\Omega, \eta_\alpha) = B^+(\Omega, \eta_\alpha) - B^-(\Omega, \eta_\alpha), \tag{5.5.17}$$

where $B^+(\Omega, \eta_\alpha)$ is as in equation (2.6.13) with the functions b, c, d and e being calculated from the density ρ^+ and elastic moduli λ^+, μ^+ of the half-space $x_3 > 0$, while $B^-(\Omega, \eta_\alpha)$, because it contains the functions $\xi_3^{+\alpha}$, $\xi_3^{+\beta}$ rather than $\xi_3^{-\alpha}$, $\xi_3^{-\beta}$, is given by a similar expression but with the signs of the radicals in (2.6.14) reversed. The complete matrix $A(\Omega, \eta_\alpha)$ retains the general form (2.6.13) and it will be convenient to define it as the right side of (2.6.13), but with the functions b, c, d and e now being calculated in the way indicated above. It follows that

$$A^{-1}(\Omega, \eta_\alpha) = \frac{1}{be-d^2}\begin{bmatrix} e-c'\eta_2^2 & c'\eta_1\eta_2 & \mathrm{i}d\eta_1 \\ c'\eta_1\eta_2 & e-c'\eta_1^2 & \mathrm{i}d\eta_2 \\ -\mathrm{i}d\eta_1 & -\mathrm{i}d\eta_2 & b \end{bmatrix}, \tag{5.5.18}$$

where

$$c' = (ce+d^2)/(b+c), \tag{5.5.19}$$

and hence by inspection that the fundamental matrix $\chi(z, \eta)$ can be constructed exactly as in § 5.2, except that $\lambda_1(p)$ is replaced by $1/\lambda_1(p)$ and $(b/e)^{\frac{1}{2}}$ is replaced by $(e/b)^{\frac{1}{2}}$. Making these replacements, and approximating $(e/b)^{\frac{1}{2}}$ by

$$(e/b)^{\frac{1}{2}} \simeq K \prod_{i=1}^{n} (1-p^2/c_i^2), \tag{5.5.20}$$

an approximate fundamental matrix $\chi(z, \eta)$ is generated, which has exactly the form (5.4.2) and whose columns can be scaled so that $\bar{\chi}(z, \eta) = \chi(z, \eta)$.

For the case $n = 0$, equation (5.4.15) yields essentially two terms. First, when $m = 0$,

$$F^{(3)}(z, \eta_\alpha) = \mathrm{i}S_0 \begin{bmatrix} \mathrm{i}\eta_1 g^o(z) \\ \mathrm{i}\eta_2 g^o(z) \\ h^e(z) \end{bmatrix} (z^2 - V^2)^{-2}, \tag{5.5.21}$$

since $F^{(3)}(z, \eta_\alpha)$ must be an even function of (z, η). Secondly, when $m = 1$,

$$F^{(3)}(z, \eta_\alpha) = (\eta_1 + \mathrm{i}\eta_2)\,(-U_0 R_0 + \mathrm{i}V_2(z, \eta)\, T_0)\,(z^2 - V^2)^{-2}, \tag{5.5.22}$$

which gives two independent possibilities, of which one is

$$F^{(3)}(z,\eta_\alpha) = \mathrm{i}\begin{bmatrix} R_0-\eta_1^2(R_0-\mathrm{i}g^{\mathrm{e}}(z)\,T_0) \\ -\eta_1\eta_2(R_0-\mathrm{i}g^{\mathrm{e}}(z)\,T_0) \\ \eta_1 h^{\mathrm{o}}(z)\,T_0 \end{bmatrix}(z^2-V^2)^{-2}. \tag{5.5.23}$$

This is obtained by taking $\frac{1}{2}(F^{(3)}-\bar{F}^{(3)})$; a similar term is obtained from $\frac{1}{2}(F^{(3)}+\bar{F}^{(3)})$, but this will not be displayed. Now (5.5.23) is derived from a Radon transform $\check{b}(p,\eta_\alpha)$ only if (5.4.17) is satisfied for $k = 0, 1$. For $k = 1$, it is satisfied identically, while the equation for $k = 0$ implies

$$\oint z^3(R_0-\mathrm{i}g^{\mathrm{e}}(z)\,T_0)\,(z^2-V^2)^{-2}\,\mathrm{d}z = 0. \tag{5.5.24}$$

Deforming the contour of integration to a large circle now shows that

$$R_0 = \mathrm{i}g^{\mathrm{e}}(\infty)\,T_0. \tag{5.5.25}$$

That R_0 and T_0 may both be real follows because $\bar{g}^{\mathrm{e}}(z) = -g^{\mathrm{e}}(z)$, by construction.

The traction vector $T(t,x_\alpha)$ is obtained by integration of (3.1.13) or, equivalently, (4.5.4); the latter gives

$$T(t,x_\alpha) = (2\pi)^{-\frac{5}{2}}\oint_{|\eta|=1}\mathrm{d}s\,\mathrm{Re}\left\{\int_{\eta_\alpha x_\alpha/t}^{\infty}\mathrm{d}\Omega\,A^{-1}(-\Omega+0\mathrm{i},\eta_\alpha)\,F^{(3)}(\Omega-0\mathrm{i},\eta_\alpha)\right\}. \tag{5.5.26}$$

This integral may be split into two parts to give

$$T(t,x_\alpha) = T(t,0)+(2\pi)^{-\frac{5}{2}}\oint_{|\eta|=1}\mathrm{d}s\,\mathrm{Re}\left\{\int_{\eta_\alpha x_\alpha/t}^{0}\mathrm{d}\Omega\,A^{-1}(-\Omega+0\mathrm{i},\eta_\alpha)\,F^{(3)}(\Omega-0\mathrm{i},\eta_\alpha)\right\}, \tag{5.5.27}$$

where
$$T(t,0) = -(2\pi)^{-\frac{5}{2}}\,\mathrm{Im}\left\{\int_0^\infty \mathrm{d}p\oint_{|\eta|=1}\mathrm{d}s\,A^{-1}(\mathrm{i}p,\eta_\alpha)\,F^{(3)}(-\mathrm{i}p,\eta_\alpha)\right\}, \tag{5.5.28}$$

having chosen the contour of integration to be the negative imaginary axis in (5.5.28).

Considering first the form (5.5.21) for $F^{(3)}(z,\eta_\alpha)$, equation (5.5.28) shows, by performing the elementary integration with respect to η, that

$$T_1(t,0) = T_2(t,0) = 0, \tag{5.5.29}$$

while
$$T_3(t,0) = -(2\pi)^{-\frac{3}{2}}S_0\int_0^\infty\frac{\mathrm{d}p[-dg^{\mathrm{o}}(\mathrm{i}p)+bh^{\mathrm{e}}(\mathrm{i}p)]}{(be-d^2)\,(V^2+p^2)^2}, \tag{5.5.30}$$

in which b, e and d are evaluated for $\Omega = \mathrm{i}p$. The 'imaginary part' sign could be dropped, as in the preceding example, because $g^{\mathrm{o}}(\mathrm{i}p)$ and $h^{\mathrm{e}}(\mathrm{i}p)$ are real. The form (5.5.28) thus corresponds to tensile loading.

By taking $F^{(3)}(z,\eta_\alpha)$ as in equation (5.5.23), it is obtained similarly that

$$T_2(t,0) = T_3(t,0) = 0,$$

$$T_1(t,0) = -(2\pi)^{-\frac{3}{2}}\frac{T_0}{2}\int_0^\infty\frac{\mathrm{i}\,\mathrm{d}p[e(g^{\mathrm{e}}(\infty)+g^{\mathrm{e}}(\mathrm{i}p))-dh^{\mathrm{o}}(\mathrm{i}p)]}{(be-d^2)\,(V^2+p^2)^2}, \tag{5.5.31}$$

showing that this corresponds to shear loading in the x_1-direction. The other term obtainable from (5.5.22) is exactly similar and corresponds to shear loading in the x_2-direction.

Various aspects of the solution could now be examined, but here attention will be restricted to the elastic arrivals as these are of interest in seismology. They are obtainable from (3.1.15) and

(2.2.15), and the appropriate expression for $F^{(3)}(z, \eta_\alpha)$. As an illustration, the P-wave arrival from a crack expanding under shear loading will be discussed in detail. This is probably the simplest arrival and yet it can still display some rather complex behaviour.

First, using (3.1.15) and the homogeneity property (A 2) of the Radon transform, we obtained that

$$\tilde{T}(\omega, \xi_\alpha) = (2\pi)^{-2}\mathrm{i}|\xi|^{-4}A^{-1}(\omega, \xi_\alpha)\,F^{(3)}(-\omega/|\xi|, \eta_\alpha). \tag{5.5.32}$$

Equation (2.2.15) now gives, for $x_3 > 0$,

$$\partial_t^2 u(t, x) \sim -\frac{4\pi x_3}{\alpha|x|^2}\mathrm{Im}\left\{B^\alpha(1, \xi_\lambda^0)\,\tilde{T}(1, \xi_\lambda^0)\left[\ln\left|t-\frac{|x|}{\alpha}\right| - \mathrm{i}\pi H\left(\frac{|x|}{\alpha}-t\right)\right]\right\}, \tag{5.5.33}$$

as $t \to |x|/\alpha$, in which $\xi_\lambda^0 = -x_\lambda/(\alpha|x|)$, $B^\alpha\tilde{T}$ is $(2\pi)^{-\frac{3}{2}}$ times the term $U^{-\alpha}b^\alpha$ that appears in equation (2.6.5) and $\alpha = \alpha_+$, the speed of P-waves in the half-space $x_3 > 0$. Now for shear loading, $F^{(3)}(z, \eta_\alpha)$ is given by equation (5.5.23) and it follows after some manipulation that

$$B^\alpha(\omega, \xi_\lambda)\,\tilde{T}(\omega, \xi_\lambda) = \frac{T_0\xi_1[2|\xi|\,\xi_3^{-\beta}(eg^{\mathrm{e}}+dh^{\mathrm{o}})+\mathrm{i}(2|\xi|^2-\omega^2/\beta^2)\,(dg^{\mathrm{e}}+bh^{\mathrm{o}})]}{(2\pi)^{\frac{7}{2}}\mu D(\omega, |\xi|)\,(be-d^2)\,(\omega^2-V^2|\xi|^2)^2}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3^{-\alpha}\end{bmatrix}, \tag{5.5.34}$$

in which b, d and e are functions of $\Omega = \omega/|\xi|$ and g^{e} and h^{o} are functions of $z = -\omega/|\xi|$. The P-wave radiation pattern produced by a shear crack now follows by substitution of (5.5.34) into (5.5.33), with $\omega = 1$ and $\xi_\lambda = -x_\lambda/(\alpha|x|)$. Since now $\xi_3^{-\alpha} = -x_3/(\alpha|x|)$, the P-wave arrival is radial, as it should be, but it can be quite complicated. It is not, for example, very simply related to the 'double couple' P-wave arrival, even if the double couple is taken at the interface, for this would be obtained by taking $b(t, x_\alpha)$ proportional to $\delta(x_1)\,\delta(x_2)$, so that $F^{(3)}(z, \eta_\alpha)$ would be proportional to z^{-4} and would not involve g^{e}, h^{o}. This contrasts with the arrival produced by a crack in a homogeneous medium, in which the arrival is just that of a double couple, modified by the factor $(\omega^2-V^2|\xi|^2)^{-2}$ (Burridge & Willis 1969). It may also be noted that, when $z = -1/|\xi|$, $g^{\mathrm{e}}(z)$ is imaginary while $h^{\mathrm{o}}(z)$ is real. Also, $D(1, |\xi|)$ is real, and so a step function arrival is obtained if b and e are imaginary and d is real. It is clear from equations (2.6.14) that this will occur if $\alpha_+|x|\,(x_\lambda x_\lambda)^{-\frac{1}{2}}$ is greater than the speed, α_- say, of P-waves in the half-space $x_3 < 0$ and, in particular, for all x (with $x_3 > 0$) if α is the larger of the two P-wave speeds. If α is the smaller of the two P-wave speeds, however, the arrival will have a logarithmic component in the region

$$x_\lambda x_\lambda > \alpha_+^2|x|^2/\alpha_-^2. \tag{5.5.35}$$

This singularity is not associated with a sharp wave front with no disturbance ahead of it, but is physically acceptable because a disturbance could have reached x before the arrival time $t = |x|/\alpha_+$ by travelling partly in the lower half-space. A 'conical wave' may thus be present in the P-wave arrival, as well as in the S-wave. The analysis of § 2.2 was directed towards finding just the direct body-wave arrivals and gives no information on the actual first motion in the region (5.5.35).

The P-wave arrival given by (5.5.33) is homogeneous of degree -1 in x and a simple way of representing it graphically is to plot a polar diagram, with the arrival as radial variable, while x ranges over the unit sphere; this gives the shape of the set of points x on which the arrival is constant. Figure 6 displays such a polar diagram, when x is in the plane $x_2 = 0$. The properties of the half space were taken as $\alpha_+ = 6.5$, $\beta_+ = 3.74$, $\alpha_- = 7.76$, $\beta_- = 4.36$ (km s^{-1}) and $\rho_+ = 2.85$, $\rho_- = 3.3$ (g cm^{-3}), characteristic of the 'Moho' interface between the basaltic layer and the upper mantle of the earth (Jeffreys 1970), and the crack speed V was taken as $V = 0.8\beta_+$. Approximations like those used in Example (5.5 (*a*)) were employed, so that

$$(e/b)^{\frac{1}{2}} \simeq 1 - p^2/c_1^2. \tag{5.5.36}$$

This gave an imaginary c_1 and, correspondingly, an imaginary k_1, so that it was necessary to scale the column $V_2(z, \eta)$ of $\chi(z, \eta)$ by i to ensure that $\chi(z, \eta) = \bar{\chi}(z, \eta)$.

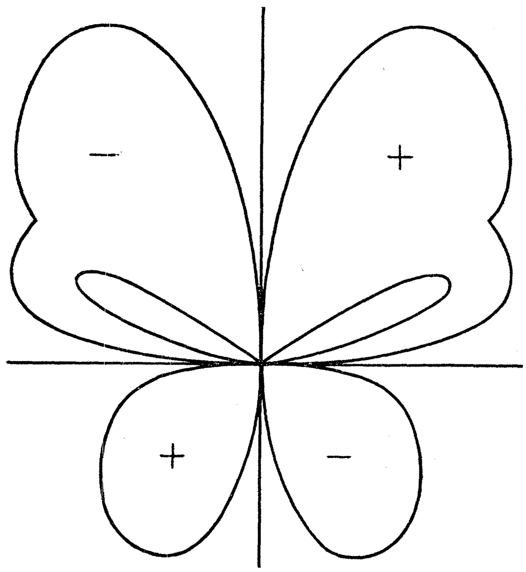

FIGURE 6. The P-wave 'radiation pattern' produced, in the plane $x_2 = 0$ by a circular crack expanding with speed $V = 2.99\ \mathrm{km\ s^{-1}}$ on the Moho interface ($x_3 = 0$), in response to an applied shear stress σ_{13}.

Both the step function and the logarithmic arrival are plotted. The corresponding arrival for a crack in a homogeneous medium is obtainable either by setting $g^e = i$, $h^o = 0$ and $d = 0$ in (5.5.34), or directly from equation (5.27) of Burridge & Willis (1969). Each give the same result, that there is only a step-function arrival, whose amplitude is proportional to

$$x_1 x_3 \{|x| [|x|^2 - V^2(x_1^2 + x_2^2)/\alpha^2]\}^{-1}.$$

This yields four lobes, rather similar to those shown in the lower half of figure 6, which was obtained from (5.5.33) by interchanging α_+ with α_- and so on, but keeping the original value of V. Thus, the significant change in the P-wave arrival occurs in the half-space with the slower wave speeds. This is true even of the step-function arrival, whose curve shows a sharp corner at the boundary of the region (5.5.35). The significance of the logarithmic arrival is exaggerated in figure 6 for ease of plotting. The step-function locus was actually obtained by plotting those values of x for which the amplitude of the step function arrival was $\pm\pi$, while the other locus was obtained by plotting values at which the amplitude of the logarithmic arrival was $\pm\frac{1}{5}$.

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Appendix. Properties of the Radon transform

This appendix is similar to those in the author's earlier papers (Willis 1971*a*, 1972) and lists the properties of the Radon transform that are used in the text.

The Radon transform $\check{f}(p,\xi)$ of a function $f(x)$ is defined as its integral along the line

$$\xi \,.\, x = p.$$

Thus,

$$\check{f}(p,\xi) = \int f(x)\,\delta(\xi\,.\,x - p)\,\mathrm{d}x. \tag{A 1}$$

It follows that $\check{f}(p,\xi)$ is even and homogeneous of degree -1, so that

$$\check{f}(\lambda p, \lambda\xi) = |\lambda|^{-1}\check{f}(p,\xi). \tag{A 2}$$

Also,

$$\int p^k \check{f}(p,\xi)\,\mathrm{d}p = \int f(x)\,\delta(\xi\,.\,x-p)\,(\xi\,.\,x)^k\,\mathrm{d}x,$$

so that the left side is a homogeneous polynomial of degree k in ξ. Equivalently, if η is the unit vector

$$\eta = \xi/|\xi|,$$

then

$$\int p^k \check{f}(p,\eta)\,\mathrm{d}p = P_k(\eta), \tag{A 3}$$

a polynomial of degree at most k in η. A further property relates to the support of $\check{f}(p, \eta)$. It follows from (A 1) that

$$f(x) = 0, \quad r > a \Rightarrow \check{f}(p, \eta) = 0, \quad |p| > a. \tag{A 4}$$

The properties (A 2), (A 3) and (A 4) have been shown by Ludwig (1966) to be sufficient for a function $\check{f}(p, \eta)$ to define a Radon transform of a function $f(x)$ whose support is the disk $r \leqslant a$.

Finally, the inversion formula for the Radon transform is quoted as

$$f(x) = -\frac{1}{4\pi^2} \oint_{|\eta|=1} \mathrm{d}s \int \frac{\check{f}(p, \eta)\,\mathrm{d}p}{(p - \eta \,.\, x + 0\mathrm{i})^2}. \tag{A 5}$$

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